Incorporating Domain Knowledge in Matching Problems via Harmonic Analysis

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Can We Accelerate the Solution of Quadratic Assignment Problem (QAP)?

We propose a new approach for doing this by learning if (a) multiple QAP instances of interest come from the same application, and (b) the correct solution for a set of such QAP instances is given.

- Parameter learning problem is too general for arbitrary y (target labels). What if the label y has a specific algebraic structure? More specifically, what if y is a candidate in the symmetric group $S_n$?

### Fourier Transform

Familiar Fourier transform provides a unifying mathematical approach to study a complicated function $f : \mathbb{R} \to \mathbb{C}$ as a sum represented by $\sum_{\lambda \in \mathbb{R}} f(\lambda) e^{i\lambda n}$, where $\lambda \in \mathbb{R}$ describe the domain of analysis, and $e^{i\lambda n}$ is the irreducible representation of $\mathbb{R}$.

![Figure: (Left) Function defined on R, (right) Fourier transform of the function.]

#### FOURIER TRANSFORM ON SYMMETRIC GROUP

For $f : S_n \to \mathbb{C}$ is the expansion of $f$ in terms of irreducible matrix representation of $S_n$, realized over conjugacy classes (partitions of n).

$$f(\rho) = \sum_{\pi \in \mathcal{P}} f(\pi) \rho(\pi)$$

Where $\rho(\pi) \in \mathbb{R}$.

Irreps Construction – Youngs Orthogonal Representation (YOR): Irreps of $S_n$ are indexed by partitions of n. A partition $\lambda$ of n is a k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of integers such that $\lambda_1 \geq \lambda_2 \geq \ldots > \lambda_k > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. A Young diagram corresponds to a specific partition $\lambda$. Specific strategy of filling in the boxes of a Young diagram: numbers increase to the right across rows and down the columns, provide a combinatorial object called Standard Young tableaux.

![Figure: (Left) Young diagrams (1,4), (3,1), (2,2), (1,1,1,1), (right) Standard Young tableaux for $\lambda = [3,1]$.]

### Braching Rule and Adapted Representation

Describes the relationship between irreducible representations of $S_n$ and those of $S_{n-1}$. A $\lambda'$ denotes $\lambda + \nu$ obtained by adding one square to $\mu + 1$ which can be shown graphically using Young diagrams. YOR is adapted to $S_n \geq S_{n-1}$ and $\rho_{\lambda'}$ is called induced representation of $S_n$. Similarly, YOR provides $\rho_{\lambda''}$.

### COSET TREE

Systematically split the entire $S_n$ using a special permutation called a contigous cycle.

### PROBLEM STATEMENT

In applications that allow various ways of extracting features (hence provide various adjacency matrices), we define our goal as follows:

- Find a match such that edge $(i, j)$ in G should be assigned to an edge $(\hat{i}, \hat{j})$ in $G'$ that is of a similar length (or weight) simultaneously in all adjacency matrices.

### OUR APPROACH

- Standard quadratic assignment objective expressed as band-limited function defined on $S_n$ called graph correlation function $f(\lambda)$

$$f(\lambda) = \frac{1}{\sqrt{\lambda_{\max}}} f(\omega)$$

where $f(\omega)$ and $f(\omega)$ are graph functions of G and $G'$. Graph function on the adjacency matrix A of a graph $G$ has useful sparsity pattern in the Fourier domain [2].

- We define graph correlation function on each feature representation as $f' = f(\omega)$ defined on $|A^T, A^M|$ where $d \in D$.

- Parameterize each $f'(\sigma)$ and write a base QAP objective for learning

$$f'(\sigma) = \sum_{d=1}^{D} \sum_{\sigma} \rho(\sigma)$$

where $\sigma \in \mathbb{R}^d$ represents parameterization.

- Learning amounts to adjusting the $\omega$ appearing in base QAP objective using the true assignments $\sigma^*$ given for each training pair $(G_m, G_m')$.

$$\omega^* = \arg \min_{\omega} \sum_{m=1}^{M} L(\sigma_m(\omega), \sigma_m') + \Omega(\omega)$$

- Note that $\sigma_m(\omega)$ itself corresponds to solving a QAP objective given a base QAP modulated by parameter $\omega$.

### ALGORITHM: LEARNING IN FOURIER SPACE

Fast Fourier transform of function $f'$ with respect to YOR:

$$f'(\lambda) = \sum_{i=1}^{\rho(\lambda)} \rho(\pi) f(\lambda) \bigoplus_{\mu, \lambda \in \mathcal{P}} f'(\mu)$$

### EXPERIMENTAL RESULTS

Task: 2D image alignment using local features [1].

Features: (1) Edge features, (2) Distance features, (3) Shape Context features.

Dataset and Setup: (1) CMU House dataset, (2) CMU Hotel dataset, (3) Silhouette dataset.

![Figure: Ground truth (green) and the learnt correspondences (red). (Left) House 27 – 97 frame. (Center) Hotel 58 – 98 frame. (Right) Shear 66 – 146 frame.]

- Fourier space bounds are defined as:

$$B_{\rho_{\lambda}} = \sum_{\rho(\mu) = \rho(\lambda)} ||f'_{\mu}(\pi)||_2$$

- Fourier space QAP solver compares Fourier space bounds at each level in coset tree.

- Jensen’s Inequality: For parameterization $f'_{\mu}(\pi) = \sum_{d=1}^{D} \omega_d f'(\mu)$, we write easy-to-optimize set of bounds,

$$||f'_{\lambda}(\pi)||_2 = \sum_{d=1}^{D} \omega_d ||f'(\mu)||_2$$

- Stochastic Gradient Descent Solver: For all examples, we minimize relative bounds between correct nodes and their incorrect siblings at each level in coset tree.

$$\sum_{k=1}^{n} \sum_{i \in \text{children}(k)} f'(\mu) - f'_{\pi}(\mu) + 1$$

- For $\Omega(\pi) = \pi^2$, each update takes the form

$$\omega_{\mu} = \omega_{\mu} - \eta (f'_{\mu}(\pi) - ||f'(\mu)||_2 + \omega_{\mu}||f'(\mu)||_2)$$

- SODA, 2010.