Incorporating Domain Knowledge in Matching Problems via Harmonic Analysis

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Can we accelerate the solution of Quadratic Assignment Problem (QAP)?

We propose a new approach for doing this by *learning* if (a) multiple QAP instances of interest come from the same application, and (b) the correct solution for a set of such QAP instances is given.

- Parameter learning problem is too general for arbitary $\mathbf{y}$ (target labels). What if the label $\mathbf{y}$ has a specific algebraic structure? More specifically, what if $\mathbf{y}$ is a candidate in the symmetric group $\mathbb{S}_n$?

Fourier Transform

Familiar Fourier transform provide a unifying mathematical approach to study a complicated function $f : \mathbb{R} \to \mathbb{C}$ as a sum represented by $\sum_{x\in\mathbb{R}} f(x)e^{2\pi i x\lambda}$, where $\lambda \in \mathbb{R}$ describe the domain of analysis, and $e^{2\pi i x\lambda}$ is the irreducible representation of $\mathbb{R}$.

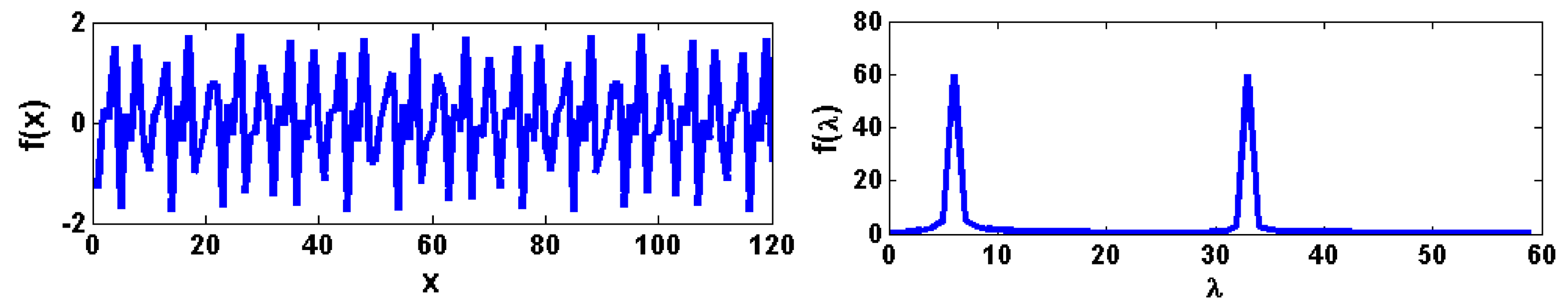


Figure: (Left) Function defined on $\mathbb{R}$, (right) Fourier transform of the function.

Fourier Transform on Symmetric Group

For $f : \mathbb{S}_n \to \mathbb{C}$ is the expansion of f in terms of *irreducible matrix representation* of $\mathbb{S}_n$, realized over conjugacy classes (partitions of n).

$$\hat{f}(\rho_\lambda) = \sum_{\sigma\in\mathbb{S}_n} f(\sigma)\rho_\lambda(\sigma) \qquad \rho_\lambda \in \mathcal{R}$$

Irreps Construction – Youngs Orthogonal Representation (YOR): Irreps of $\mathbb{S}_n$ are indexed by partitions of n. A partition λ of n is a k-tuple $(\lambda_1, \cdots, \lambda_k)$ of integers such that $\lambda_1 \geq \cdots \geq \lambda_k > 0$ and $\lambda_1 + \cdots + \lambda_k = n$. A *Young diagram* corresponds to a specific partition λ. Specific strategy of filling in the boxes of a Young diagram: numbers increase to the right across rows and down the columns, provide a combinatorial object called *Standard Young tableau*.

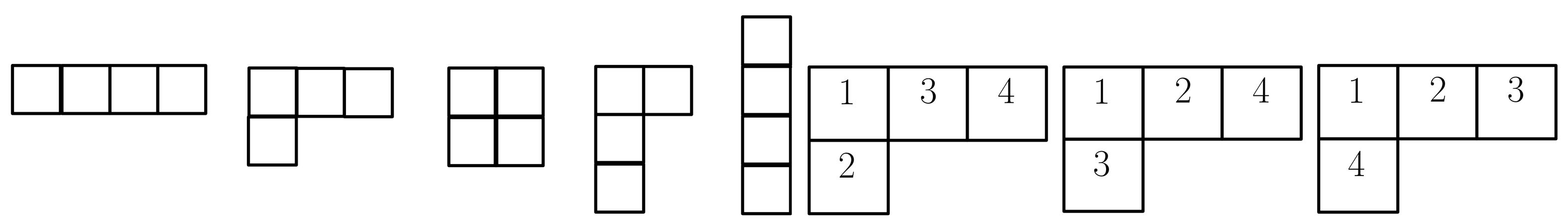
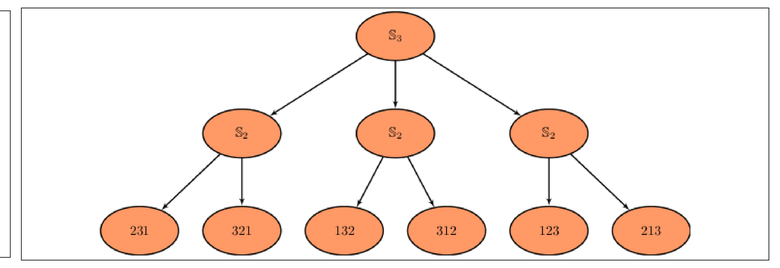


Figure: (Left) Young diagrams $\{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}$, (right) Standard Young tableau for $\lambda = (3,1)$

Branching Rule and Adapted Representation: Describes the relationship between irreducible representations of $\mathbb{S}_n$ and those of $\mathbb{S}_{n-1}$. $\lambda\uparrow^n$ denote $\lambda \vdash n$ obtained by adding one square to $\mu \vdash n-1$ which can be shown graphically using Young diagrams. YOR is adapted to $\mathbb{S}_n \geq \mathbb{S}_{n-1}$, and $\rho_{\lambda\uparrow^n}$ is called induced representation of $\mathbb{S}_n$. Similary, YOR provide $\rho_{\mu\downarrow_{n-1}}$.

Coset Tree: Systematically split the entire $\mathbb{S}_n$ using a special permutation called *contiguous cycle*.

$$[[i,j]](k) = \begin{cases} k+1 & \text{for } k = i, i+1, \cdots, j-1 \\ i & \text{for } k = j \\ k & \text{otherwise} \end{cases}$$



Properties of Cosets:

- Order of a coset is same as that of subgroup $\mathbb{S}_{n-k}$
- Any two left (right) cosets are either disjoint or the same

Problem Statement

In applications that allow various ways of extracting features (hence provide various adjacency matrices), we define our **goal** as follows:

- Find a match such that edge (i,j) in G should be assigned to an edge (i',j') in G' that is of a similar length (or weight) simultaneously in all adjacency matrices.

Our Approach

- Standard quadratic assignment objective expressed as band-limited function defined on $\mathbb{S}_n$ called Graph correlation,
$$\hat{f}(\lambda) = \frac{1}{(n-2)!}\hat{f}_A(\lambda)\hat{f}_{A'}(\lambda)$$
where f_A and $f_{A'}$ are graph functions of G and G'. Graph function on the adjacency matrix A of a graph G has useful sparsity pattern in the Fourier domain [2].
- We define graph correlation function on each feature representation as f^d defined on (A^d, A'^d) where $d \in D$.
- Parameterize each $f^d(\sigma)$ and write a base QAP objective for learning
$$f^\omega(\sigma) = \sum_{d=1}^{D} f^d_{\omega_d}(\sigma)$$
where $\omega \in \mathbb{R}^d$ represents parameterization.
- Learning amounts to adjusting the ω appearing in base QAP objective using the *true* assignments σ^* given for each training pair (G_m, G'_m).
$$\omega^* = \arg\min_{\omega} \sum_{m=1}^{M} L(\hat{\sigma}_m(\omega), \sigma^*_m) + \Omega(\omega)$$
- **Note** that $\sigma_m(\omega)$ itself corresponds to solving a QAP objective given a base QAP modulated by parameter ω.

Algorithm: Learning in Fourier space

Fast Fourier transform of function f^ω with respect to YOR:

$$\hat{f}^\omega(\lambda) = \sum_{i=1}^{n} \rho_\lambda([[i,n]]) \bigoplus_{\mu\in\lambda\downarrow n-1} \hat{f}^\omega_i(\mu)$$

- Fourier space bounds are defined as:
$$B_{n\to i} = \sum_{\mu\vdash n-1} \|\hat{f}^\omega_i(\mu)\|_*$$
Fourier space QAP solver compares Fourier space bounds at each level in coset tree.
- *Jensen's Inequality:* For parameterization $\hat{f}^\omega_i(\mu) = \sum_{d=1}^{D}\omega_d\hat{f}^d_i(\mu)$, we write easy-to-optimize set of bounds,
$$\|\hat{f}^\omega_i(\mu)\|_* = \|\sum_{d=1}^{D}\omega_d\hat{f}^d_i(\mu)\|_* \leq \sum_{d=1}^{D}\omega_d\|\hat{f}^d_i(\mu)\|_*$$
- Stochastic Gradient Descent Solver: For all examples, we minimize relative bounds between correct nodes and their incorrect siblings at each level in coset tree.
$$\sum_{k=1}^{n}\sum_{i\in\text{children}((n-k+1)^*)}\left[\hat{f}^\omega_i(\mu) - \hat{f}^\omega_{i^*_{n-k}}(\mu) + 1\right]^+$$
- For $\Omega(w) = \frac{\nu}{2}\|\omega\|_2^2$, each update takes the form
$$\omega_d \leftarrow \omega_d - \eta\begin{cases} \|\hat{f}^d_i(\mu)\|_* - \|\hat{f}^d_{i^*_{n-k}}(\mu)\|_* + \frac{\nu}{M\mathcal{O}(n^2)}\omega_d \\ \frac{\nu}{M\mathcal{O}(n^2)}\omega_d \end{cases}$$

Experimental Results

Task: 2D image alignment using local features [1].
Features: (1) Edge–featutres, (2) Distance–features, (3) Shape Context–features.
Dataset and Setup: (1) CMU House dataset, (2) CMU Hotel dataset, (3) Silhouette dataset.

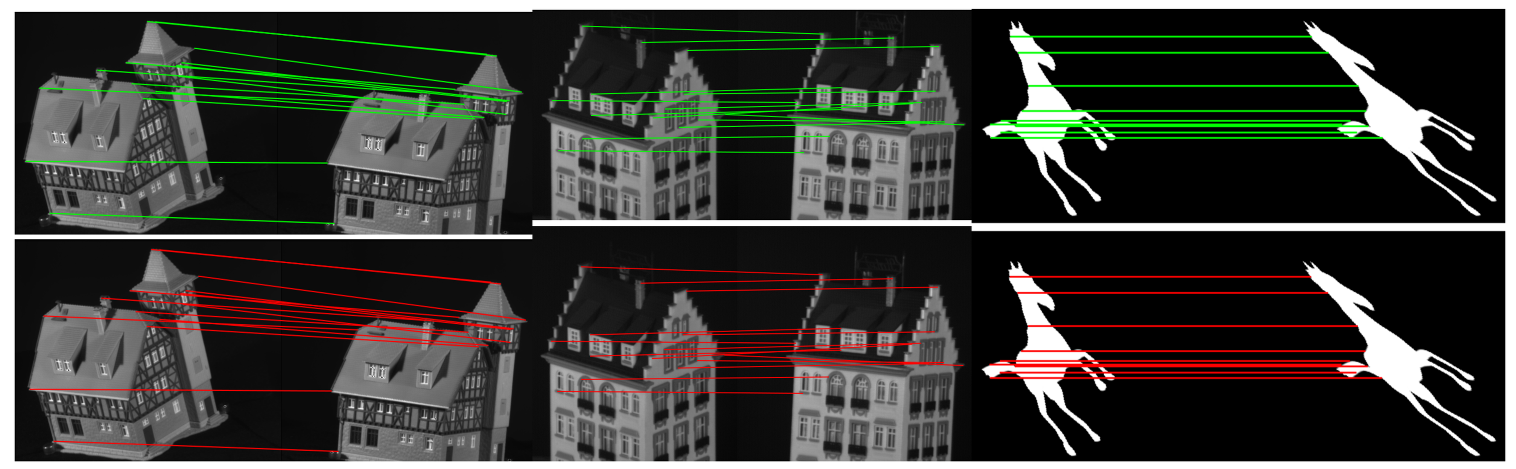


Figure: Ground truth (green) and the learnt correspondences (red). (Left) House 27 – 97 frame. (Center) Hotel 58 – 98 frame. (Right) Shear 66 – 146 frame.

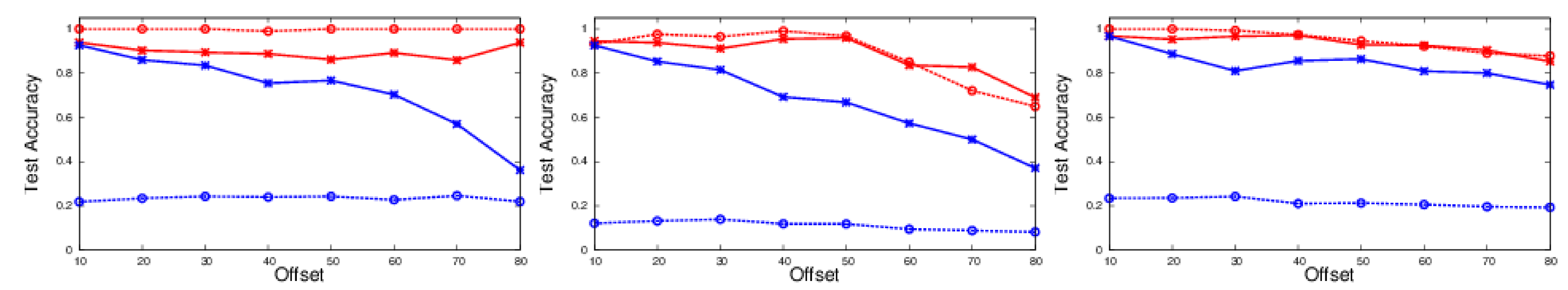


Figure: (Red) learning and (blue) no-learning. (Dashed) Delaunay, distance and 5 uninformative features. (Bold) Delaunay, distance and shape context features. (Left) House, (Center) Hotel, and (Right) Silhouette.

- T. Caetano, J. J. McAuley, L. Cheng, Q. V. Le, and A. Smola, *Learning graph matching*, PAMI **31** (2009), no. 6, 1048–1058.
- R. Kondor, *A Fourier space algorithm for solving quadratic assignment problems*, SODA, 2010.