Asymmetric encryption

CS642: Computer Security

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Announcements

- Please turn in project proposals today
- Any time is fine
- Email them directly to me
  - subject line should include CS642 project proposal
  - I will reply to your email so you know I got it
  - (If you don't hear back by Thursday let me know)

Asymmetric encryption

Basic setting

The RSA algorithm

PKCS #1 encryption

Digital signing & public-key infrastructure

Hybrid encryption

Bank customer

TLS handshake for RSA transport

Bank

Pick random Nc

ClientHello, MaxVer, Nc, Ciphers/CompMethods

ServerHello, Ver, Ns, SessionID, Cipher/CompMethod

Pick random Ns

CERT = (pk of bank, signature over it)

Check CERT using CA public verification key

C

Pick random PMS
C <- E(pk,PMS)

PMS <- D(sk,C)

ChangeCipherSpec,
{ Finished, PRF(MS, “Client finished” || H(transcript)) }

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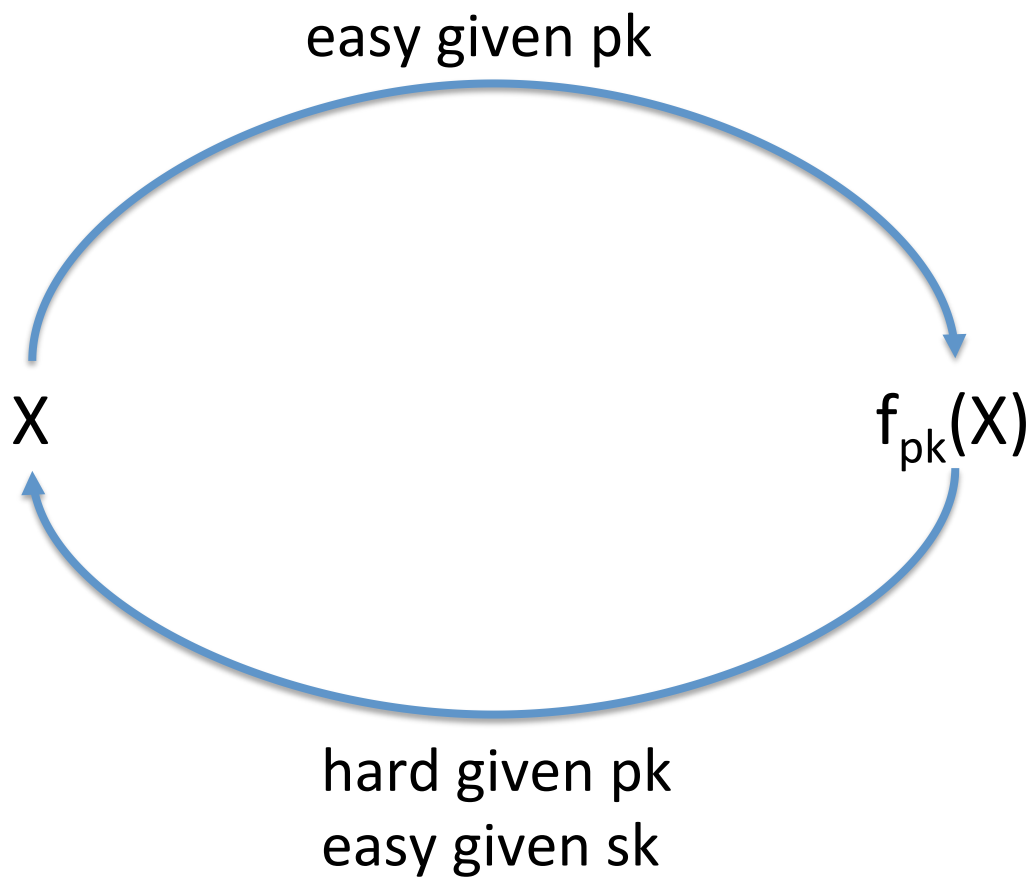
Bracket notation means contents encrypted

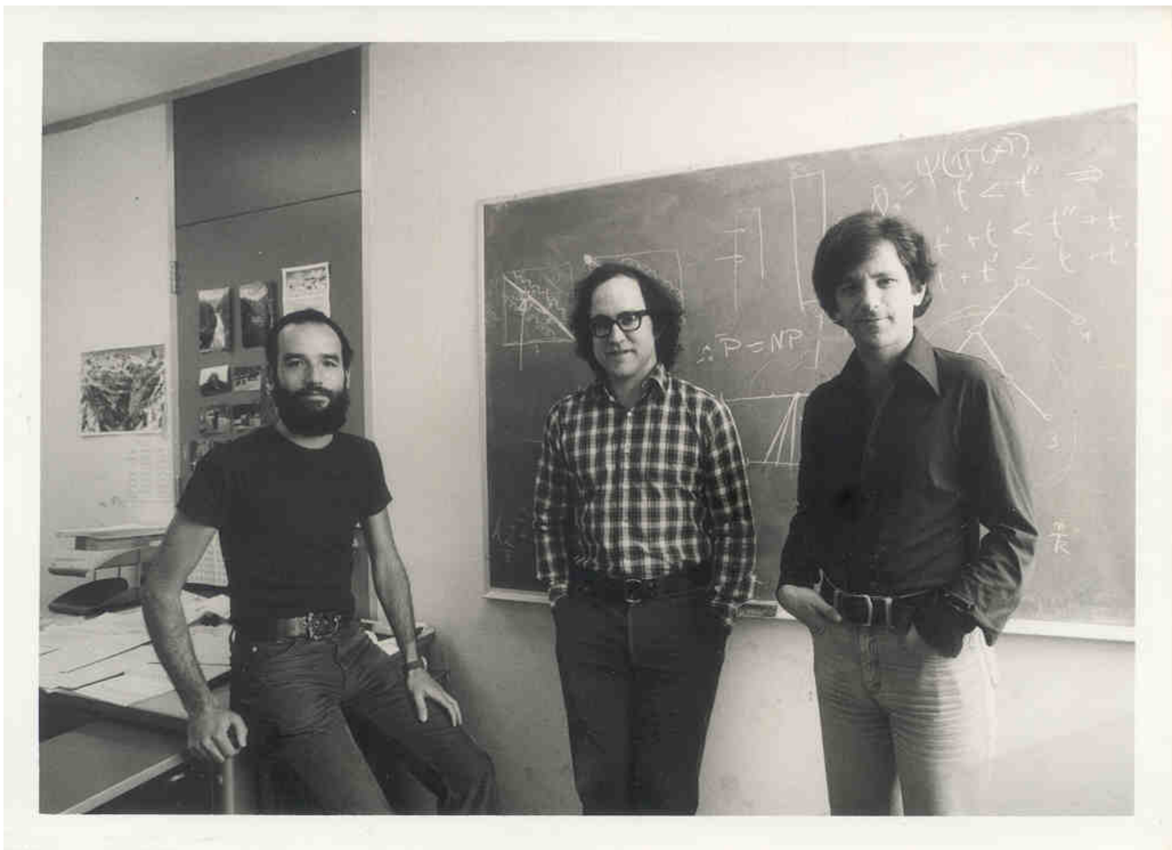
MS <- PRF(PS, “master secret” || Nc || Ns)

Trapdoor functions help us build PKE



The RSA trapdoor function

- Rivest, Shamir, Adleman 1978
- Garnered them a Turing award



RSA math

p and q be large prime numbers
N = pq
N is called the modulus

p = 7, q = 13, gives N = 91

p = 17, q = 53, gives N = 901

RSA math

p and q be large prime numbers

$N = pq$

N is called the modulus

$\mathbf{Z}_N = \{0,1,2,3,\ldots, N-1\}$

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

The size of a set S is denoted by $|S|$

$\gcd(X,Y) = 1$ if greatest common divisor of X,Y is 1

RSA math

$\mathbf{Z}_N^* = \{\, i \mid \gcd(i,N) = 1 \,\}$

N = 13 $\mathbf{Z}_{13}^* = \{ 1,2,3,4,5,6,7,8,9,10,11,12 \}$

N = 15 $\mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$

Def. $\phi(N) = |\mathbf{Z}_N^*|$ (This is Euler's totient function)

$\phi(13) = 12$

$\phi(15) = 8$

$\mathbf{Z}_{\phi(15)}^* = \mathbf{Z}_8^* = \{ 1,3,5,7 \}$

RSA math

$\mathbf{Z}_N^* = \{\, i \mid \gcd(i,N) = 1 \,\}$

$\mathbf{Z}_N^*$ is a group under modular multiplication

Fact. For any a,N with N > 0, there exists unique q,r such that

$$a = Nq + r \qquad \text{and} \qquad 0 \le r < N$$

17 mod 15 = 2

Def. $a \bmod N = r \in \mathbf{Z}_N$

105 mod 15 = 0

Def. $a \equiv b \pmod{N}$ iff $(a \bmod N) = (b \bmod N)$

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

$\mathbf{Z}_N^*$ is a group under modular multiplication

$$\mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$$

$2 \bullet 7 \equiv 14 \pmod{15}$

$4 \bullet 8 \equiv 2 \pmod{15}$

Closure: for any $a,b \in \mathbf{Z}_N$ $\quad a \bullet b \bmod N \in \mathbf{Z}_N$

Def. $\quad a^i \bmod N = \underbrace{a \bullet a \bullet a \bullet \ldots \bullet a}_{i \text{ times}} \bmod N$

RSA math

$\mathbf{Z}_N^* = \{\, i \mid \gcd(i,N) = 1 \,\}$

Claim: Suppose $e,d \in \mathbf{Z}_{\phi(N)}^*$ satisfying $ed \bmod \phi(N) = 1$ then for any $x \in \mathbf{Z}_N^*$ we have that

$$(x^e)^d \bmod N = x$$

$$\begin{aligned}(x^e)^d \bmod N &= x^{(ed \bmod \phi(N))} \bmod N \\ &= x^1 \bmod N \\ &= x \bmod N\end{aligned}$$

First equality is by Euler's Theorem

RSA math

$\mathbf{Z}_N^*$ = { i | gcd(i,N) = 1 }

Claim: Suppose $e,d \in \mathbf{Z}_{\phi(N)}^*$ satisfying ed mod $\phi(N) = 1$ then for any $x \in \mathbf{Z}_N^*$ we have that

$$(x^e)^d \bmod N = x$$

$\mathbf{Z}_{15}^*$ = { 1,2,4,7,8,11,13,14 } $\mathbf{Z}_{\phi(15)}^*$ = { 1,3,5,7 }

e = 3 , d = 3 gives ed mod 8 = 1

x	1	2	4	7	8	11	13	14
x^3 mod 15	1	8	4	13	2	11	7	14
y^3 mod 15	1	2	4	7	8	11	13	14

RSA admits a trapdoor permutation

$pk = (N,e)$ $sk = (N,d)$ with $ed \bmod \phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

$$\begin{tikzcd}[column sep=huge]
X \arrow[r, bend left, "\text{easy given } N,e"] & f_{pk}(X) \arrow[l, bend left, "\substack{\text{hard given } N,e \\ \text{easy given } N,d}"]
\end{tikzcd}$$

RSA admits a trapdoor permutation

pk = (N,e) sk = (N,d) with ed mod $\phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

But how do we find suitable N,e,d ?

If p,q distinct primes and N = pq then $\phi(N) = (p-1)(q-1)$

Why?

$$\begin{aligned}\phi(N) &= |\{1,\dots,N-1\}| - |\{ip : 1 \le i \le q-1\}| - |\{iq : 1 \le i \le p-1\}| \\ &= N-1 - (q-1) - (p-1) \\ &= pq - p - q + 1 \\ &= (p-1)(q-1)\end{aligned}$$

RSA admits a trapdoor permutation

$pk = (N,e)$ $sk = (N,d)$ with $ed \bmod \phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

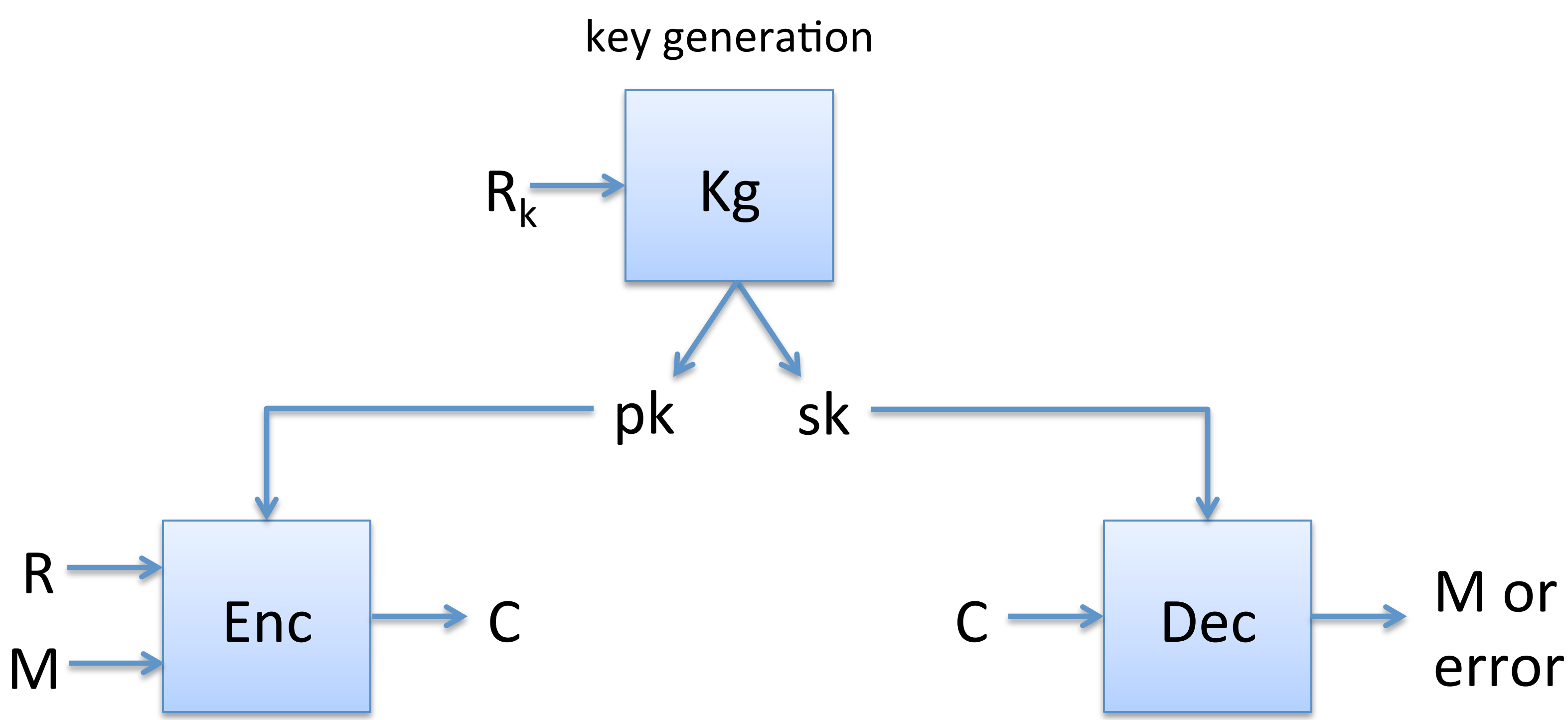
But how do we find suitable N,e,d ?

If p,q distinct primes and $N = pq$ then $\phi(N) = (p-1)(q-1)$

Given $\phi(N)$, choose $e \in \mathbf{Z}_{\phi(15)}$ and calculate

$d = e^{-1} \bmod \phi(N)$

Public-key encryption



C is a ciphertext

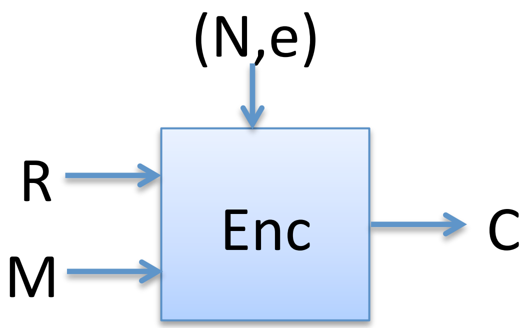
Correctness: D(sk , E(pk,M,R)) = M with probability 1 over randomness used

PKCS #1 RSA encryption

Kg outputs (N,e),(N,d) where $|N|_8 = n$

Let $B = \{0,1\}^8 / \{00\}$ be set of all bytes except 00

Want to encrypt messages of length $|M|_8 = m$

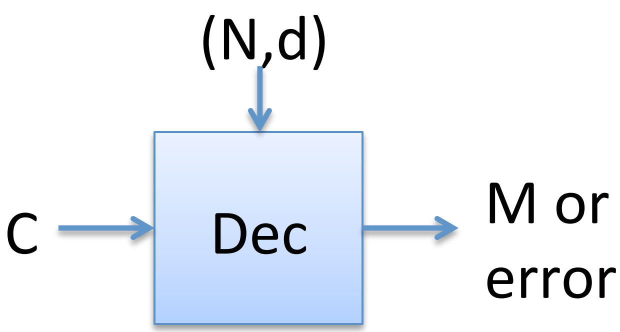


<u>Enc((N,e), M, R)</u>

pad = first n - m - 2 bytes from R that are in B

$X = 00 \,||\, 02 \,||\, pad \,||\, 00 \,||\, M$

Return $X^e \bmod N$



<u>Dec((N,d), C)</u>

$X = C^d \bmod N$; $aa||bb||w = X$

If $(aa \neq 00)$ or $(bb \neq 02)$ or $(00 \notin w)$

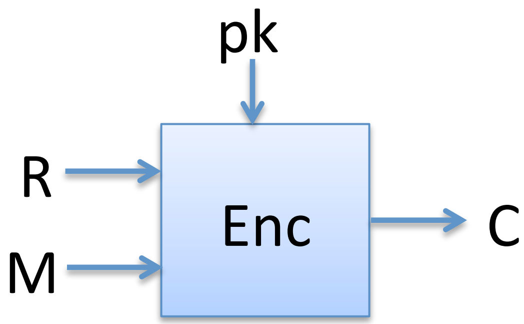
Return error

$pad \,||\, 00 \,||\, M = w$

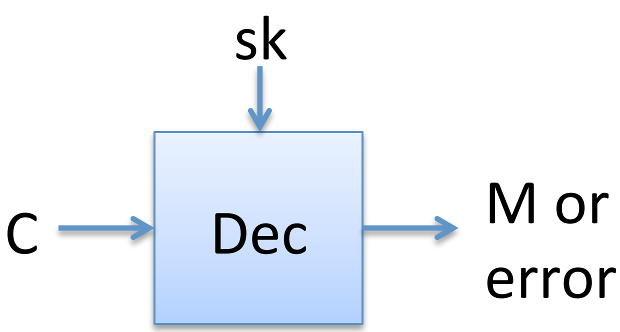
Return M

Hybrid encryption

Kg outputs (pk,sk)



<u>Enc(pk, M, R)</u>
K||R1||R2 = R
C1 = Enc(pk,K,R1)
C2 = Enc(K,M,R2)
Return (C1,C2)



<u>Dec(sk, (C1,C2))</u>
K = Dec(sk,C1)
M = Dec(K,C2)
Return M

TLS handshake for RSA transport

Bank customer

Bank

Bank customer	Message	Bank
Pick random Nc	ClientHello, MaxVer, Nc, Ciphers/CompMethods →	
	← ServerHello, Ver, Ns, SessionID, Cipher/CompMethod	Pick random Ns
Check CERT using CA public verification key	← CERT = (pk of bank, signature over it)	
Pick random PMS C <- E(pk,PMS)	C →	PMS <- D(sk,C)
	ChangeCipherSpec, { Finished, PRF(MS, “Client finished” || H(transcript)) } →	
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Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?

Inverting RSA : given N, e, y find x such that $x^e \equiv y \pmod{N}$

EASY — because $f^{-1}(y) = y^d \bmod N$

Know d

EASY — because $d = e^{-1} \bmod \varphi(N)$

Know $\varphi(N)$

EASY — because $\varphi(N) = (p-1)(q-1)$

Know p, q

? — Learning p,q from N is the factoring problem

Know N

We don't know if inverse is true, whether inverting RSA implies ability to factor

Factoring composites

- What is p,q for N = 901?

```
Factor(N):
for i = 2 , … ,  sqrt(N) do
     if N mod i = 0 then
          p = i
          q = N / p
          Return (p,q)
```

Woops… we can always factor

But not always efficiently:
Run time is sqrt(N)

$\mathcal{O}(\text{sqrt}(N)) = \mathcal{O}(e^{0.5 \ln(N)})$

Factoring composites

Algorithm	**Time to factor N**
Naïve	$\mathcal{O}(e^{0.5 \ln(N)})$
Quadratic sieve (QS)	$\mathcal{O}(e^{c})$ $c = d (\ln N)^{1/2} (\ln \ln N)^{1/2}$
Number Field Sieve (NFS)	$\mathcal{O}(e^{c})$ $c = 1.92 (\ln N)^{1/3} (\ln \ln N)^{2/3}$

Factoring records

Algorithm	**Year**	**Algorithm**	**Time**
RSA-400	1993	QS	830 MIPS years
RSA-478	1994	QS	5000 MIPS years
RSA-515	1999	NFS	8000 MIPS years
RSA-768	2009	NFS	~2.5 years

RSA-x is an RSA challenge modulus of size about x bits

Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?
  - Pick |N| > 1024 and factoring will fail
  - Active attacks?

Bleichanbacher attack

I've just learned some information about $C_1^d \bmod N$

C_1 →

← padding error?

C_2 →

← padding error?

…

Dec((N,d), C)
X = C^d mod N   ; aa||bb||w = X
If (aa ≠ 00) or (bb ≠ 02) or (00   w)
    Return error
pad || 00 || M = w
Return M

We can take a target C and decrypt it using a sequence of chosen ciphertexts C_1, … , C_q where q ~ 1 million

Response to this attack

- Ad-hoc fix: Don't leak whether padding was wrong or not
  - This is harder than it looks (timing attacks)
- Better:
  - use chosen-ciphertext secure encryption
  - OAEP is common choice

Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?
  - Pick |N| > 1024 and factoring will fail
  - Active attacks?
    - Some implementations seem ok
  - Man-in-the-middle: replace (N,e) with our own key

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Pick random Ns

← ServerHello, Ver, Ns, SessionID, Cipher/CompMethod

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C →

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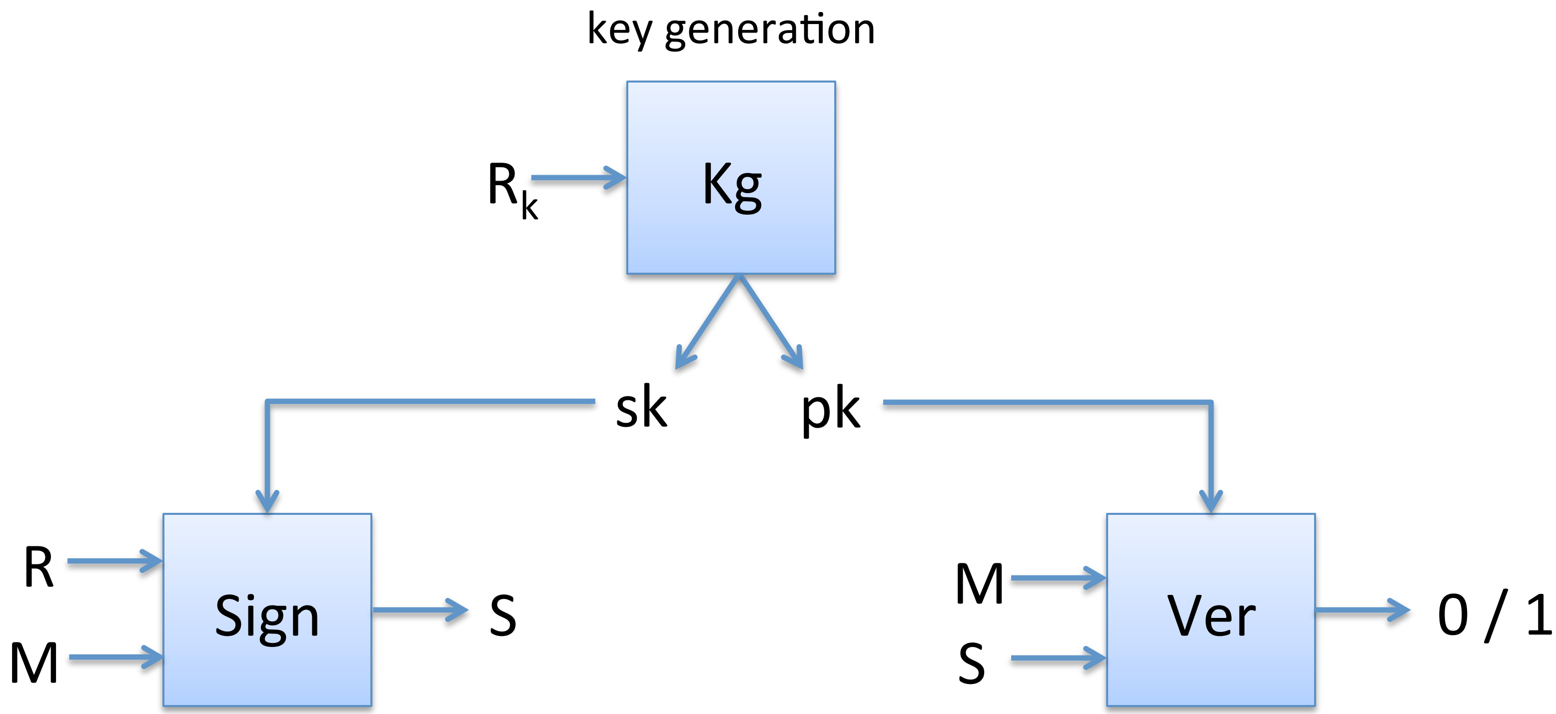
Digital signatures

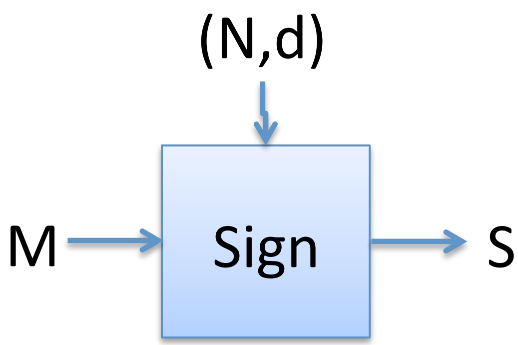


Anyone with public key can verify a signature
Only holder of secret key should be able to generate a signature

Full Domain Hash RSA

Kg outputs pk = (N,e) , sk = (N,d)

H is a hash function

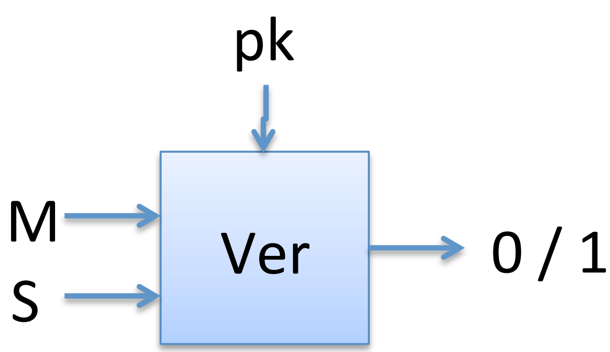


<u>Sign((N,d), M)</u>

$X = 00 \,||\, H(1||M) \,||\, \ldots \,||\, H(k||M)$

$S = X^d \bmod N$

Return S



<u>Ver((N,e), M, S)</u>

$X = S^e \bmod N$

$X' = 00 \,||\, H(1||M) \,||\, \ldots \,||\, H(k||M)$

If $X = X'$ then

    Return 1

Return 0

Certificate Authorities and Public-key Infrastructure

(pk,sk)

VeriSign

pk

M = (pk’,data)
S = Sign(sk,M)

Give me a certificate for pk’, please

S

http://amazon.com

pk’, data, S

M = (pk’,data)
If Ver(pk,M,S) then
trust pk’

(pk’,sk’)

This prevents man-in-the-middle (MitM) attacks