Problems with Gauss-Seidel

- All the form factors are required before any image can be generated
- Reducing the number of form factors requires reducing the number of patches, which severely impacts quality
- We desire a *progressive* solution, that starts with a rough approximation and refines it

Radiosity Eqn to Energy Eqn

$$B_i = E_i + \rho_i \sum_{j \neq i} B_j F_{ij} \quad \text{for} \quad 1 \leq i < N$$

Rewrite this equation in terms of energy values per patch (instead of per unit area)

$$\beta_i = \varepsilon_i + \rho_i \sum_{j \neq i} \beta_j F_{ji} \quad \text{with} \quad \beta_i = A_i B_i, \varepsilon_i = A_i E_i$$

$$\boldsymbol{K\beta} = \boldsymbol{\varepsilon} \quad \text{with} \quad \boldsymbol{K}_{ij} = \delta_{ij} - \rho_i F_{ji}$$

Relaxation and Residuals

- *Relaxation methods* start with an initial guess, $\boldsymbol{\beta}^{(0)}$, and perform a sequence of relaxation steps, each resulting in a new $\boldsymbol{\beta}^{(k)}$
- Define $\boldsymbol{r}^{(k)} = \boldsymbol{\varepsilon} - \boldsymbol{K\beta}^{(k)}$, the *residual*
- At each step, relaxation methods zero one element of the residual (e.g. Gauss-Seidel zeros each one in turn)

Southwell Relaxation

- Southwell relaxation zeros the *largest* residual at each step

$$0 = r_i^{(k+1)} = \varepsilon_i - \sum_{j=1}^{N} K_{ij}\beta_j^{(k+1)}$$

$$\forall j \neq i, \beta_j^{(k+1)} = \beta_j^{(k)}$$

$$\beta_i^{(k+1)} = \frac{1}{K_{ii}}\left(\varepsilon_i - \sum_{j\neq i} K_{ij}\beta_j^{(k)}\right)$$

$$r_j^{(k+1)} = r_j^{(k)} - \frac{K_{ji}}{K_{ii}} r_i^{(k)}$$

Southwell Summary

- Each patch has two components: energy, β_i^k, and undistributed energy, r_i^k
- Start with some β_i^0 and hence r_i^0
- At each step k+1:
  - Choose the i with maximum residual
  - Update β_i^{k+1} and $r_i^{k+1} = 0$
  - Update all the r_j^{k+1}

Physical Interpretation

- Assume that all the initial patch energies are 0
- Then the initial residuals are the amounts of energy to be emitted by each patch
- Each step redistributes the residual according to:

$$r_j^{(k+1)} = r_j^{(k)} + \rho_j F_{ij} r_i^{(k)}$$

- So, each patch gets its own share of the residual that is shot, according to the form factors

Gathering and Shooting

- Gauss-Seidel "gathers" radiosity from every patch to a specific patch:

$$B_i^{(k+1)} = E_i + \rho_i \sum_{j \neq i} B_j^{(k)} F_{ij}$$

- Southwell "shoots" energy from one patch onto all the other patches

$$r_j^{(k+1)} = r_j^{(k)} + \rho_j F_{ij} r_i^{(k)}$$

Progressive Refinement

- After any number of iterations, an estimate of each patch's final energy can be obtained by: $B_i^{(k)} = \frac{\beta_i^{(k)} + r_i^{(k)}}{A_i}$
- These intermediate results can be displayed as the algorithm proceeds, giving faster feedback

Ambient Correction

- Progressive radiosity images look dark at first, because shooters hold onto their energy until it's their turn.
- An ambient correction can be added *to the display only*:

$$R = \frac{1}{1-\rho_{avg}}, U = \sum_{j=1}^{N} r_j^{(k)}, \Delta B_i = \frac{\rho_i RU}{\sum_{i=1}^{N} A_i}$$

Capturing Detail

- Finer patches are required around shadow boundaries and fast changes in radiosity
- Halving the linear dimension of all patches results in 16x more work
- Fine patches are needed as *receivers*, but not as *emitters*
- *Substructuring* addresses this

Substructuring

- Break each patch into smaller *elements* for the purposes of receiving energy
- Divide each patch, P_i into m_i elements, denoted p_q for $1 \leq q < m_i$.
- Radiosity for element p_q is:

$$B_{\bar{q}} = E_{\bar{q}} + \rho_{\bar{q}} \sum_{j=1}^{N} F_{p_q, P_j} B_j$$

Substructuring (more)

- Patch radiosity is area weighted radiosity of elements:

$$B_i = \frac{1}{A_i} \sum_{q=1}^{m_i} B_{\bar{q}} A_{\bar{q}}$$

- Substituting, and assuming elements of a patch have equal exitance and reflectance:

$$B_i = E_i + \rho_i \sum_{j=1}^{N} \left[\frac{1}{A_i} \sum_{q=1}^{m_i} F_{p_q, P_j} A_{\bar{q}} \right] B_j$$

- Using area averaged form factor

Solution Structure

- Find element to patch form factors (M×N)
- Combine to form patch to patch form factors (N × N)
- Solve for patch radiosities (no worse than regular Gauss-Seidel)
- Compute element radiosities

Substructuring and Progressive Refinement

- Can do progressive refinement with substructuring
- Shoot energy from patches to elements
- But, form factor computations are poor, because the form factor from a large patch to a small element is needed
- Leads to visible errors in images

Adaptive Subdivision

- Instead of fixing the elements and patches, allow elements to be subdivided based on how the solution proceeds
- After each Gauss-Seidel step, look at the radiosity gradient, and break elements with high gradients. Then redo the computation with the new elements
- With progressive radiosity, can subdivide differently for each shot, interpolating and averaging as necessary