Chapter 1: Probability Theory
Lecture 1: Measurable space, measure and probability

Random experiment: uncertainty in outcomes

Sample space

\( \Omega \): sample (or outcome) space: a set containing all possible outcomes

Definition 1.1

Let \( \mathcal{F} \) be a collection of subsets of a sample space \( \Omega \).
\( \mathcal{F} \) is called a \( \sigma \)-field (or \( \sigma \)-algebra) iff it has the following properties.

(i) The empty set \( \emptyset \in \mathcal{F} \).
(ii) If \( A \in \mathcal{F} \), then the complement \( A^c \in \mathcal{F} \).
(iii) If \( A_i \in \mathcal{F}, \; i = 1, 2, \ldots \), then their union \( \bigcup A_i \in \mathcal{F} \).

\( (\Omega, \mathcal{F}) \) is a measurable space if \( \mathcal{F} \) is a \( \sigma \)-field on \( \Omega \)

Discussion

\( \mathcal{F} \) is a collection (set) of sets
Two trivial examples: \( \mathcal{F} \) contains \( \emptyset \) and \( \Omega \) only;
\( \mathcal{F} \) contains all subsets of \( \Omega \) (power set)
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Two trivial examples: \( \mathcal{F} \) contains \( \emptyset \) and \( \Omega \) only;
\( \mathcal{F} \) contains all subsets of \( \Omega \) (power set)
Why do we need to consider other $\sigma$-field?

We may be interested in a particular collection of sets e.g., $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$, where $A \subset \Omega$

$\mathcal{C}$ = a collection of sets of interest

$\mathcal{C}$ may not be a $\sigma$-field

$\sigma(\mathcal{C})$: the smallest $\sigma$-field containing $\mathcal{C}$ (the $\sigma$-field generated by $\mathcal{C}$)

$\sigma(\mathcal{C}) = \mathcal{C}$ if $\mathcal{C}$ itself is a $\sigma$-field

$\Gamma = \{\mathcal{F} : \mathcal{F}$ is a $\sigma$-field on $\Omega$ and $\mathcal{C} \subset \mathcal{F}\}$

$\sigma(\mathcal{C}) = \bigcap_{\mathcal{F} \in \Gamma}\mathcal{F}$

$\sigma(\{A\}) = \sigma(\{A, A^c\}) = \sigma(\{A, \Omega\}) = \sigma(\{A, \emptyset\}) = \{\emptyset, A, A^c, \Omega\}$

Borel $\sigma$-field

$\mathbb{R}^k$: the $k$-dimensional Euclidean space ($\mathbb{R}^1 = \mathbb{R}$ is the real line)

$\mathcal{O}$: the collection of all open sets

$\mathcal{B}^k = \sigma(\mathcal{O})$: the Borel $\sigma$-field on $\mathbb{R}^k$

$\mathcal{B}^k = \sigma(\mathcal{C})$, $\mathcal{C}$ is the collection of all closed sets

$C \in \mathcal{B}^k$, $\mathcal{B}_C = \{C \cap B : B \in \mathcal{B}^k\}$ is the Borel $\sigma$-field on $C$
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Measure
Length, area, volume, ...

Definition 1.2.
Let \((\Omega, \mathcal{F})\) be a measurable space. A set function \(\nu\) defined on \(\mathcal{F}\) is called a measure iff it has the following properties.

(i) \(0 \leq \nu(A) \leq \infty\) for any \(A \in \mathcal{F}\).

(ii) \(\nu(\emptyset) = 0\).

(iii) If \(A_i \in \mathcal{F}, i = 1, 2, \ldots,\) and \(A_i\)'s are disjoint, i.e., \(A_i \cap A_j = \emptyset\) for any \(i \neq j\), then

\[
\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i).
\]

\((\Omega, \mathcal{F}, \nu)\) is a measure space if \(\nu\) is a measure on a \(\sigma\)-field on \(\Omega\).
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Conventions

- For any $x \in \mathbb{R}$, $\infty + x = \infty$, $x \infty = \infty$ if $x > 0$, $x \infty = -\infty$ if $x < 0$, and $0 \infty = 0$;
- $\infty + \infty = \infty$;
- $\infty^a = \infty$ for any $a > 0$;
- $\infty - \infty$ or $\infty/\infty$ is not defined

Probability measure

If $\nu(\Omega) = 1$, then $\nu$ is a probability measure.
We usually use notation $P$ instead of $\nu$.
$(\Omega, \mathcal{F}, P)$ is a probability space if $P$ is a probability measure on a $\sigma$-field on $\Omega$.

Important examples of measures

- Measures take $\infty$ as its value:

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\nu(A) = \begin{cases} 
\infty & A \in \mathcal{F}, A \neq \emptyset \\
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Important examples of measures

- **Point mass:**
  Let \( x \in \Omega \) be a fixed point.

  \[
  \delta_x(A) = \begin{cases} 
  c & x \in A \\
  0 & x \notin A.
  \end{cases}
  \]

- **Counting measure:**
  Let \( \Omega \) be a sample space, \( \mathcal{F} \) the collection of all subsets, and \( \nu(A) \) the number of elements in \( A \in \mathcal{F} \) (\( \nu(A) = \infty \) if \( A \) contains infinitely many elements).
  Then \( \nu \) is a measure on \( \mathcal{F} \) and is called the *counting measure*.

- **Lebesgue measure:**
  There is a unique measure \( m \) on \( (\mathbb{R}, \mathcal{B}) \) that satisfies 
  \( m([a, b]) = b - a \) for every finite interval \( [a, b], \ -\infty < a \leq b < \infty \).
  This is called the *Lebesgue measure*.
  If we restrict \( m \) to the measurable space \( ([0, 1], \mathcal{B}[0,1]) \), then \( m \) is a probability measure.
Proposition 1.1 (Properties of measures)

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space.

1. (Monotonicity). If $A \subset B$, then $\nu(A) \leq \nu(B)$.

2. (Subadditivity). For any sequence $A_1, A_2, \ldots$, 
   \[ \nu \left( \bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \nu(A_i). \]

3. (Continuity). If $A_1 \subset A_2 \subset A_3 \subset \cdots$ (or $A_1 \supset A_2 \supset A_3 \supset \cdots$ and $\nu(A_1) < \infty$), then 
   \[ \nu \left( \lim_{n \to \infty} A_n \right) = \lim_{n \to \infty} \nu(A_n), \]
   where 
   \[ \lim_{n \to \infty} A_n = \bigcup_{i=1}^{\infty} A_i \quad \left( \text{or} = \bigcap_{i=1}^{\infty} A_i \right) . \]
Cumulative distribution function

Let $P$ be a probability measure on $(\mathbb{R}, \mathcal{B})$. The *cumulative distribution function* (c.d.f.) of $P$ is defined to be

$$F(x) = P((\neg \infty, x]), \quad x \in \mathbb{R}$$

**Proposition 1.2 (Properties of c.d.f.’s)**

(i) Let $F$ be a c.d.f. on $\mathbb{R}$.

(a) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$;
(b) $F(\infty) = \lim_{x \to \infty} F(x) = 1$;
(c) $F$ is nondecreasing, i.e., $F(x) \leq F(y)$ if $x \leq y$;
(d) $F$ is right continuous, i.e., $\lim_{y \to x, y > x} F(y) = F(x)$.

(ii) Suppose a real-valued function $F$ on $\mathbb{R}$ satisfies (a)-(d) in part (i). Then $F$ is the c.d.f. of a unique probability measure on $(\mathbb{R}, \mathcal{B})$. 
Cumulative distribution function)

Let \( P \) be a probability measure on \((\mathbb{R}, \mathcal{B})\). The \textit{cumulative distribution function} (c.d.f.) of \( P \) is defined to be

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Product space

\[ \mathcal{I} = \{1, \ldots, k\}, \ k \text{ is finite or } \infty \]

\( \Gamma_i, \ i \in \mathcal{I}, \) are some sets

\[ \prod_{i \in \mathcal{I}} \Gamma_i = \Gamma_1 \times \cdots \times \Gamma_k = \{(a_1, \ldots, a_k) : a_i \in \Gamma_i, i \in \mathcal{I}\} \]

\( \mathbb{R} \times \mathbb{R} = \mathbb{R}^2, \ \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 \)

Product \( \sigma \)-field

Let \((\Omega_i, \mathcal{F}_i), \ i \in \mathcal{I},\) be measurable spaces

\( \prod_{i \in \mathcal{I}} \mathcal{F}_i \) is not necessarily a \( \sigma \)-field

\( \sigma (\prod_{i \in \mathcal{I}} \mathcal{F}_i) \) is called the product \( \sigma \)-field on the product space \( \prod_{i \in \mathcal{I}} \Omega_i \)

\( (\prod_{i \in \mathcal{I}} \Omega_i, \sigma (\prod_{i \in \mathcal{I}} \mathcal{F}_i)) \) is denoted by \( \prod_{i \in \mathcal{I}} (\Omega_i, \mathcal{F}_i) \)

Example: \( \prod_{i=1,\ldots,k}(\mathbb{R},\mathcal{B}) = (\mathbb{R}^k,\mathcal{B}^k) \)
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**Example:** \( \prod_{i=1, \ldots, k} (\mathbb{R}, \mathcal{B}) = (\mathbb{R}^k, \mathcal{B}^k) \)
Consider a rectangle \([a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2\).

The usual area of \([a_1, b_1] \times [a_2, b_2]\) is

\[(b_1 - a_1)(b_2 - a_2) = m([a_1, b_1])m([a_2, b_2])\]

Is \(m([a_1, b_1])m([a_2, b_2])\) the same as the value of a measure defined on the product \(\sigma\)-field?

**\(\sigma\)-finite**

A measure \(\nu\) on \((\Omega, \mathcal{F})\) is said to be \(\sigma\)-finite iff there exists a sequence \(\{A_1, A_2, \ldots\}\) such that \(\bigcup A_i = \Omega\) and \(\nu(A_i) < \infty\) for all \(i\).

Any finite measure (such as a probability measure) is clearly \(\sigma\)-finite.

The Lebesgue measure on \(\mathbb{R}\) is \(\sigma\)-finite, since \(\mathbb{R} = \bigcup A_n\) with \(A_n = (-n, n), \ n = 1, 2, \ldots\).

The counting measure in is \(\sigma\)-finite if and only if \(\Omega\) is countable.

The measure \(\nu(A) = \infty\) unless \(A = \emptyset\) is not \(\sigma\)-finite.
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Proposition 1.3 (Product measure theorem)

Let \((\Omega_i, \mathcal{F}_i, \nu_i), \ i = 1, \ldots, k\), be measure spaces with \(\sigma\)-finite measures, where \(k \geq 2\) is an integer. Then there exists a unique \(\sigma\)-finite measure on the product \(\sigma\)-field \(\sigma(\mathcal{F}_1 \times \cdots \times \mathcal{F}_k)\), called the product measure and denoted by \(\nu_1 \times \cdots \times \nu_k\), such that

\[
\nu_1 \times \cdots \times \nu_k(A_1 \times \cdots \times A_k) = \nu_1(A_1) \cdots \nu_k(A_k)
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for all \(A_i \in \mathcal{F}_i, \ i = 1, \ldots, k\).

The joint c.d.f.

Let \(P\) be a probability measure on \((\mathbb{R}^k, \mathcal{B}^k)\). The joint c.d.f. of \(P\) is defined by

\[
F(x_1, \ldots, x_k) = P((-\infty, x_1] \times \cdots \times (-\infty, x_k] ), \quad x_i \in \mathbb{R}
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There is a one-to-one correspondence between probability measures and joint c.d.f.'s on \(\mathbb{R}^k\).
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There is a one-to-one correspondence between probability measures and joint c.d.f.'s on \(\mathbb{R}^k\).
Marginal c.d.f.

If $F(x_1, \ldots, x_k)$ is a joint c.d.f., then

$$F_i(x) = \lim_{x_j \to \infty, j = 1, \ldots, i-1, i+1, \ldots, k} F(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k)$$

is a c.d.f. and is called the $i$th marginal c.d.f.

The c.d.f. and product measure

- Marginal c.d.f.’s are determined by their joint c.d.f.
- But a joint c.d.f. cannot be determined by $k$ marginal c.d.f.’s.
- If

$$F(x_1, \ldots, x_k) = F_1(x_1) \cdots F_k(x_k), \quad (x_1, \ldots, x_k) \in \mathbb{R}^k,$$

then the probability measure corresponding to $F$ is the product measure $P_1 \times \cdots \times P_k$ with $P_i$ being the probability measure corresponding to $F_i$. 

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