Example 1.12.

Let $X$ be a random variable on $(\Omega, \mathcal{F}, P)$ whose c.d.f. $F_X$ has a Lebesgue p.d.f. $f_X$ and $F_X(c) < 1$, where $c$ is a fixed constant. Let $Y = \min\{X, c\}$, i.e., $Y$ is the smaller of $X$ and $c$.

Note that $Y^{-1}((\cdot, x]) = \Omega$ if $x \geq c$ and $Y^{-1}((\cdot, x]) = X^{-1}((\cdot, x])$ if $x < c$.

Hence $Y$ is a random variable and the c.d.f. of $Y$ is

$$F_Y(x) = \begin{cases} 1 & x \geq c \\ F_X(x) & x < c. \end{cases}$$

This c.d.f. is discontinuous at $c$, since $F_X(c) < 1$.

Thus, it does not have a Lebesgue p.d.f.

It is not discrete either.

Does $P_Y$, the probability measure corresponding to $F_Y$, have a p.d.f. w.r.t. some measure?
Example 1.12 (continued)

Consider the point mass probability measure on \((\mathbb{R}, \mathcal{B})\):

\[
\delta_c(A) = \begin{cases} 
1 & c \in A \\
0 & c \notin A 
\end{cases} \quad A \in \mathcal{B}
\]

Then \(P_Y \ll m + \delta_c\), where \(m\) is the Lebesgue measure, and the p.d.f. of \(P_Y\) is

\[
f_Y(x) = \frac{dP_Y}{d(m + \delta_c)}(x) = \begin{cases} 
0 & x > c \\
1 - F_X(c) & x = c \\
f_X(x) & x < c.
\end{cases}
\]

To show this, it suffices to show that

\[
\int_{(-\infty, x]} f_Y(t) d(m + \delta_c) = P_Y((-\infty, x]) \quad \text{for any } x \in \mathbb{R}
\]

(why?)
Example 1.12 (continued)

For $x < c$,

$$
\int_{(-\infty, x]} f_Y(t) d(m + \delta_c) = \int_{(-\infty, x]} f_X(t) dm + \int_{(-\infty, x]} f_X(t) \delta_c
$$

$$
= \int_{(-\infty, x]} f_X(t) dm = P_X((-\infty, x]) = P_Y((-\infty, x])
$$

For $x \geq c$,

$$
\int_{(-\infty, x]} f_Y(t) d(m + \delta_c) = \int_{(-\infty, c]} f_Y(t) d(m + \delta_c)
$$

$$
= \int_{(-\infty, c]} f_X(t) d(m + \delta_c) + \int_{\{c\}} [1 - F_X(c)] d(m + \delta_c)
$$

$$
= \int_{(-\infty, c]} f_X(t) dm + \int_{\{c\}} [1 - F_X(c)] d\delta_c
$$

$$
= F_X(c) + [1 - F_X(c)] = 1 = P_Y((-\infty, x])
$$
Example 1.14.


Since $Y^{-1}((-\infty, x])$ is empty if $x < 0$ and equals $Y^{-1}([0, x]) = X^{-1}([\sqrt{-x}, \sqrt{x}])$ if $x \geq 0$, the c.d.f. of $Y$ is

$$F_Y(x) = P \circ Y^{-1}((-\infty, x])$$
$$= P \circ X^{-1}([\sqrt{-x}, \sqrt{x}])$$
$$= F_X(\sqrt{x}) - F_X(-\sqrt{x})$$

if $x \geq 0$ and $F_Y(x) = 0$ if $x < 0$.

Clearly, the Lebesgue p.d.f. of $F_Y$ is

$$f_Y(x) = \frac{1}{2\sqrt{x}}[f_X(\sqrt{x}) + f_X(-\sqrt{x})]1_{(0, \infty)}(x).$$
Example 1.14 (continued)

In particular, if

\[ f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \]

which is the Lebesgue p.d.f. of the standard normal distribution \( N(0, 1) \), then

\[ f_y(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2} I_{(0, \infty)}(x), \]

which is the Lebesgue p.d.f. for the chi-square distribution \( \chi_1^2 \) (Table 1.2). This is actually an important result in statistics.
Proposition 1.8

Let $X$ be a random $k$-vector with a Lebesgue p.d.f. $f_X$ and let $Y = g(X)$, where $g$ is a Borel function from $(\mathbb{R}^k, \mathcal{B}^k)$ to $(\mathbb{R}^k, \mathcal{B}^k)$. Let $A_1, \ldots, A_m$ be disjoint sets in $\mathcal{B}^k$ such that $\mathbb{R}^k - (A_1 \cup \cdots \cup A_m)$ has Lebesgue measure 0 and $g$ on $A_j$ is one-to-one with a nonvanishing Jacobian, i.e., the determinant $\text{Det}(\partial g(x)/\partial x) \neq 0$ on $A_j$, $j = 1, \ldots, m$. Then $Y$ has the following Lebesgue p.d.f.:

$$f_Y(x) = \sum_{j=1}^m \left| \text{Det}(\partial h_j(x)/\partial x) \right| f_X(h_j(x)),$$

where $h_j$ is the inverse function of $g$ on $A_j$, $j = 1, \ldots, m$.

In Example 1.14, $A_1 = (-\infty, 0)$, $A_2 = (0, \infty)$, $g(x) = x^2$, $h_1(x) = -\sqrt{x}$, $h_2(x) = \sqrt{x}$, and $|dh_j(x)/dx| = 1/(2\sqrt{x})$. 
Proposition 1.8

Let $X$ be a random $k$-vector with a Lebesgue p.d.f. $f_X$ and let $Y = g(X)$, where $g$ is a Borel function from $(\mathbb{R}^k, \mathcal{B}^k)$ to $(\mathbb{R}^k, \mathcal{B}^k)$. Let $A_1, \ldots, A_m$ be disjoint sets in $\mathcal{B}^k$ such that $\mathbb{R}^k - (A_1 \cup \cdots \cup A_m)$ has Lebesgue measure 0 and $g$ on $A_j$ is one-to-one with a nonvanishing Jacobian, i.e., the determinant $\text{Det}(\partial g(x)/\partial x) \neq 0$ on $A_j$, $j = 1, \ldots, m$. Then $Y$ has the following Lebesgue p.d.f.:

$$f_Y(x) = \sum_{j=1}^{m} |\text{Det}(\partial h_j(x)/\partial x)| f_X(h_j(x)),$$

where $h_j$ is the inverse function of $g$ on $A_j$, $j = 1, \ldots, m$.

In Example 1.14, $A_1 = (-\infty, 0)$, $A_2 = (0, \infty)$, $g(x) = x^2$, $h_1(x) = -\sqrt{x}$, $h_2(x) = \sqrt{x}$, and $|d h_j(x)/dx| = 1/(2\sqrt{x})$. 
Example 1.15

Let $X = (X_1, X_2)$ be a random 2-vector having a joint Lebesgue p.d.f. $f_X$. Consider first the transformation $g(x) = (x_1, x_1 + x_2)$.

Using Proposition 1.8, one can show that the joint p.d.f. of $g(X)$ is

$$f_{g(X)}(x_1, y) = f_X(x_1, y - x_1),$$

where $y = x_1 + x_2$ (note that the Jacobian equals 1).

The marginal p.d.f. of $Y = X_1 + X_2$ is then

$$f_Y(y) = \int f_X(x_1, y - x_1) dx_1.$$

In particular, if $X_1$ and $X_2$ are independent, then

$$f_Y(y) = \int f_{X_1}(x_1) f_{X_2}(y - x_1) dx_1.$$
Example 1.15 (continued)

Next, consider the transformation \( h(x_1, x_2) = (x_1/x_2, x_2) \), assuming that \( X_2 \neq 0 \) a.s.

Using Proposition 1.8, one can show that the joint p.d.f. of \( h(X) \) is

\[
f_{h(X)}(z, x_2) = |x_2| f_X(zx_2, x_2),
\]

where \( z = x_1/x_2 \).

The marginal p.d.f. of \( Z = X_1/X_2 \) is

\[
f_Z(z) = \int |x_2| f_X(zx_2, x_2) \, dx_2.
\]

In particular, if \( X_1 \) and \( X_2 \) are independent, then

\[
f_Z(z) = \int |x_2| f_{X_1}(zx_2) f_{X_2}(x_2) \, dx_2.
\]
Example 1.16A (F-distribution)

Let $X_1$ and $X_2$ be independent random variables having the chi-square distributions $\chi^2_{n_1}$ and $\chi^2_{n_2}$ (Table 1.2), respectively.

The p.d.f. of $Z = X_1 / X_2$ is

$$f_Z(z) = \frac{z^{n_1/2-1} I_{(0,\infty)}(z)}{2^{(n_1+n_2)/2} \Gamma(n_1/2) \Gamma(n_2/2)} \int_0^{\infty} x^{(n_1+n_2)/2-1} e^{-(1+z)x_2/2} \, dx_2$$

$$= \frac{\Gamma[(n_1 + n_2)/2]}{\Gamma(n_1/2) \Gamma(n_2/2)} \frac{z^{n_1/2-1}}{(1 + z)(n_1+n_2)/2} I_{(0,\infty)}(z)$$

Using Proposition 1.8, one can show that the p.d.f. of

$$Y = (X_1 / n_1) / (X_2 / n_2) = (n_2 / n_1) Z$$

is the p.d.f. of the F-distribution $F_{n_1,n_2}$ given in Table 1.2.
Example 1.16B (t-distribution)

Let $U_1$ be a random variable having the standard normal distribution $N(0, 1)$ and $U_2$ a random variable having the chi-square distribution $\chi^2_{\upsilon}$. Using the same argument, one can show that if $U_1$ and $U_2$ are independent, then the distribution of $T = U_1 / \sqrt{U_2 / \upsilon}$ is the t-distribution $t_\upsilon$ given in Table 1.2.

Noncentral chi-square distribution

Let $X_1, \ldots, X_\upsilon$ be independent random variables and $X_i = N(\mu_i, \sigma^2)$. The distribution of $Y = (X_1^2 + \cdots + X_\upsilon^2) / \sigma^2$ is called the noncentral chi-square distribution and denoted by $\chi^2_{\upsilon} (\delta)$, where $\delta = (\mu_1^2 + \cdots + \mu_\upsilon^2) / \sigma^2$ is the noncentrality parameter. $\chi^2_{\upsilon} (\delta)$ with $\delta = 0$ is called a central chi-square distribution.

It can be shown (exercise) that $Y$ has the following Lebesgue p.d.f.:

$$e^{-\delta/2} \sum_{j=0}^{\infty} \frac{(\delta/2)^j}{j!} f_{2j+\upsilon}(x)$$

where $f_\upsilon(x)$ is the Lebesgue p.d.f. of the chi-square distribution $\chi^2_\upsilon$. 
Example 1.16B (t-distribution)

Let $U_1$ be a random variable having the standard normal distribution $N(0, 1)$ and $U_2$ a random variable having the chi-square distribution $\chi_n^2$. Using the same argument, one can show that if $U_1$ and $U_2$ are independent, then the distribution of $T = U_1 / \sqrt{U_2/n}$ is the t-distribution $t_n$ given in Table 1.2.

Noncentral chi-square distribution

Let $X_1, \ldots, X_n$ be independent random variables and $X_i = N(\mu_i, \sigma^2)$. The distribution of $Y = (X_1^2 + \cdots + X_n^2)/\sigma^2$ is called the noncentral chi-square distribution and denoted by $\chi_n^2(\delta)$, where 
\[
\delta = (\mu_1^2 + \cdots + \mu_n^2)/\sigma^2
\]
is the noncentrality parameter. $\chi_n^2(\delta)$ with $\delta = 0$ is called a central chi-square distribution. It can be shown (exercise) that $Y$ has the following Lebesgue p.d.f.:
\[
e^{-\delta/2} \sum_{j=0}^{\infty} \frac{(\delta/2)^j}{j!} f_{2j+n}(x)
\]
where $f_k(x)$ is the Lebesgue p.d.f. of the chi-square distribution $\chi_k^2$. 
Noncentral chi-square distribution

If $Y_1, ..., Y_k$ are independent random variables and $Y_i$ has the noncentral chi-square distribution $\chi^2_{n_i}(\delta_i)$, $i = 1, ..., k$, then $Y = Y_1 + \cdots + Y_k$ has the noncentral chi-square distribution $\chi^2_{n_1 + \cdots + n_k}(\delta_1 + \cdots + \delta_k)$.

Noncentral t-distribution and F-distribution will be introduced in discussion session

Theorem 1.5. (Cochran’s theorem)

Suppose that $X = N_n(\mu, I_n)$ and

$$X^\tau X = X^\tau A_1 X + \cdots + X^\tau A_k X,$$

where $I_n$ is the $n \times n$ identity matrix and $A_i$ is an $n \times n$ symmetric matrix with rank $n_i$, $i = 1, ..., k$.

A necessary and sufficient condition that $X^\tau A_i X$ has the noncentral chi-square distribution $\chi^2_{n_i}(\delta_i)$, $i = 1, ..., k$, and $X^\tau A_i X$’s are independent is $n = n_1 + \cdots + n_k$, in which case $\delta_i = \mu^\tau A_i \mu$ and $\delta_1 + \cdots + \delta_k = \mu^\tau \mu$. 
Noncentral chi-square distribution

If $Y_1, \ldots, Y_k$ are independent random variables and $Y_i$ has the noncentral chi-square distribution $\chi^2_{n_i}(\delta_i)$, $i = 1, \ldots, k$, then $Y = Y_1 + \cdots + Y_k$ has the noncentral chi-square distribution $\chi^2_{n_1+\cdots+n_k}(\delta_1 + \cdots + \delta_k)$.

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Noncentral chi-square distribution
If $Y_1, \ldots, Y_k$ are independent random variables and $Y_i$ has the noncentral chi-square distribution $\chi^2_{n_i}(\delta_i), i = 1, \ldots, k$, then $Y = Y_1 + \cdots + Y_k$ has the noncentral chi-square distribution $\chi^2_{n_1 + \cdots + n_k}(\delta_1 + \cdots + \delta_k)$.

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Suppose that $X = N_n(\mu, I_n)$ and
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A necessary and sufficient condition that $X^\tau A_i X$ has the noncentral chi-square distribution $\chi^2_{n_i}(\delta_i), i = 1, \ldots, k$, and $X^\tau A_i X$’s are independent is $n = n_1 + \cdots + n_k$, in which case $\delta_i = \mu^\tau A_i \mu$ and $\delta_1 + \cdots + \delta_k = \mu^\tau \mu$. 