Lecture 6: Convergence modes and relationships

Notation

$c = (c_1,...,c_k) \in \mathscr{R}^k$, $\|c\|_r = (\sum_{j=1}^k |c_j|^r)^{1/r}$, $r > 0$.
If $r \geq 1$, then $\|c\|_r$ is the L_r-distance between 0 and c.
When $r = 2$, $\|c\| = \|c\|_2 = \sqrt{c^\tau c}$.

Definition 1.8 (Covergence modes)

Let $X, X_1, X_2, \ldots$ be random k-vectors defined on a probability space.

(i) We say that the sequence $\{X_n\}$ converges to X almost surely (a.s.) and write $X_n \to_{a.s.} X$ iff $\lim_{n\to\infty} X_n = X$ a.s.

(ii) We say that $\{X_n\}$ converges to X in probability and write $X_n \to_p X$ iff, for every fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} P(\|X_n - X\| > \varepsilon) = 0.$$

(iii) We say that $\{X_n\}$ converges to X in L_r (or in rth moment) with a fixed $r > 0$ and write $X_n \to_{L_r} X$ iff

$$\lim_{n\to\infty} E\|X_n - X\|_r^r = 0$$

(iv) Let F, F_n, $n = 1, 2, \ldots$, be c.d.f.'s on $\mathscr{R}^k$ and P, P_n, $n = 1, \ldots$, be their corresponding probability measures.
We say that $\{F_n\}$ converges to F weakly (or $\{P_n\}$ converges to P weakly) and write $F_n \to_w F$ (or $P_n \to_w P$) iff, for each continuity point x of F,

$$\lim_{n\to\infty} F_n(x) = F(x).$$

We say that $\{X_n\}$ converges to X in distribution (or in law) and write $X_n \to_d X$ iff $F_{X_n} \to_w F_X$.

Remarks

- $\to_{a.s.}$, $\to_p$, $\to_{L_r}$: How close is between X_n and X as $n \to \infty$?
- $F_{X_n} \to_w F_X$: F_{X_n} is close to F_X
  but X_n and X may not be close (they may be on different spaces)

Example 1.26.

Let $\theta_n = 1 + n^{-1}$ and X_n be a random variable having the exponential distribution $E(0, \theta_n)$ (Table 1.2), $n = 1, 2, \ldots$.
Let X be a random variable having the exponential distribution $E(0, 1)$.

For any $x > 0$, as $n \to \infty$,

$$F_{X_n}(x) = 1 - e^{-x/\theta_n} \to 1 - e^{-x} = F_X(x)$$

Since $F_{X_n}(x) \equiv 0 \equiv F_X(x)$ for $x \le 0$, we have shown that $X_n \to_d X$.
$X_n \to_p X$?

- Need further information about the random variables X and X_n.
- We consider two cases in which different answers can be obtained.

Case 1

Suppose that $X_n \equiv \theta_n X$ (then X_n has the given c.d.f.).
$X_n - X = (\theta_n - 1)X = n^{-1}X$, which has the c.d.f.

$$(1 - e^{-nx})I_{[0,\infty)}(x).$$

Then, $X_n \to_p X$ because, for any $\varepsilon > 0$,

$$P(|X_n - X| \ge \varepsilon) = e^{-n\varepsilon} \to 0$$

(In fact, by Theorem 1.8(v), $X_n \to_{a.s.} X$)

Also, $X_n \to_{L_p} X$ for any $p > 0$, because

$$E|X_n - X|^p = n^{-p} EX^p \to 0$$

Case 2

Suppose that X_n and X are independent random variables.
Since p.d.f.'s for X_n and $-X$ are $\theta_n^{-1} e^{-x/\theta_n} I_{(0,\infty)}(x)$ and $e^x I_{(-\infty,0)}(x)$, respectively, we have

$$P(|X_n - X| \le \varepsilon) = \int_{-\varepsilon}^{\varepsilon} \int \theta_n^{-1} e^{-x/\theta_n} e^{y-x} I_{(0,\infty)}(x) I_{(-\infty,x)}(y) dxdy,$$

which converges to (by the dominated convergence theorem)

$$\int_{-\varepsilon}^{\varepsilon} \int e^{-x} e^{y-x} I_{(0,\infty)}(x) I_{(-\infty,x)}(y) dxdy = 1 - e^{-\varepsilon}.$$

Thus,

$$P(|X_n - X| \ge \varepsilon) \to e^{-\varepsilon} > 0$$

for any $\varepsilon > 0$ and, therefore, $X_n \to_p X$ does not hold.

Proposition 1.16 (Pólya's theorem)

If $F_n \to_w F$ and F is continuous on $\mathscr{R}^k$, then

$$\lim_{n\to\infty} \sup_{x\in\mathscr{R}^k} |F_n(x) - F(x)| = 0.$$

This proposition implies the following useful result:
If $F_n \to_w$ a continuous F and $c_n \in \mathscr{R}^k$ with $c_n \to c$, then

$$F_n(c_n) \to F(c).$$

Lemma 1.4

For random k-vectors $X, X_1, X_2, \ldots$ on a probability space, $X_n \to_{a.s.} X$ iff for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} \{\|X_m - X\| > \varepsilon\}\right) = 0.$$

Proof

It can be verified that

$$\bigcap_{j=1}^{\infty} A_j = \{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}, \quad A_j = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\|X_m - X\| \le j^{-1}\}$$

By Proposition 1.1(iii, continuity),

$$\begin{aligned} P(A_j) &= \lim_{n\to\infty} P\left(\bigcap_{m=n}^{\infty} \{\|X_m - X\| \le j^{-1}\}\right) \\ &= 1 - \lim_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} \{\|X_m - X\| > j^{-1}\}\right) \end{aligned}$$

$P\left(\bigcup_{m=n}^{\infty}\{\|X_m - X\| > \varepsilon\}\right) \to 0$ for every $\varepsilon > 0$ iff $P(A_j) = 1$ for every j, which is equivalent to $P(\cap_{j=1}^{\infty} A_j) = 1$ (i.e., $X_n \to_{a.s.} X$), because

$$P(A_j) \ge P\left(\bigcap_{j=1}^{\infty} A_j\right) = 1 - P\left(\bigcup_{j=1}^{\infty} A_j^c\right) \ge 1 - \sum_{j=1}^{\infty} P(A_j^c)$$

Lemma 1.5 (Borel-Cantelli lemma)

Let A_n be a sequence of events in a probability space and

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

(i) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup_n A_n) = 0$.

(ii) If $A_1, A_2, \ldots$ are pairwise independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\limsup_n A_n) = 1$.

Proof of Lemma 1.5 (i)

By Proposition 1.1,

$$P\left(\limsup_{n\to\infty} A_n\right) = \lim_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} A_m\right) \le \lim_{n\to\infty} \sum_{m=n}^{\infty} P(A_n) = 0$$

where the last equality follows from the condition

$$\sum_{n=1}^{\infty} P(A_n) < \infty.$$

Proof of Lemma 1.5 (ii)

We prove the case of independent A_n's.
See Chung (1974, pp. 76-78) for the pairwise independence A_n's.

$$P\left(\limsup_{n\to\infty} A_n\right) = \lim_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} A_m\right) = 1 - \lim_{n\to\infty} P\left(\bigcap_{m=n}^{\infty} A_m^c\right)$$

$$\prod_{m=n}^{n+k} P(A_m^c) = \prod_{m=n}^{n+k} [1 - P(A_m)] \le \prod_{m=n}^{n+k} \exp\{-P(A_m)\} = \exp\left\{-\sum_{m=n}^{n+k} P(A_m)\right\}$$

$(1 - t \le e^{-t} = \exp\{t\})$.

Letting $k \to \infty$,

$$\prod_{m=n}^{\infty} P(A_m^c) = \lim_{k\to\infty} \prod_{m=n}^{n+k} P(A_m^c) \le \exp\left\{-\sum_{m=n}^{\infty} P(A_m)\right\} = 0.$$

Hence,

$$\lim_{n\to\infty} P\left(\bigcap_{m=n}^{\infty} A_m^c\right) = \lim_{n\to\infty} \prod_{m=n}^{\infty} P(A_m^c) = 0.$$

The notion of $O(\cdot)$, $o(\cdot)$, and stochastic $O(\cdot)$ and $o(\cdot)$

In calculus, two sequences of real numbers, $\{a_n\}$ and $\{b_n\}$, satisfy

- $a_n = O(b_n)$ iff $|a_n| \le c|b_n|$ for all n and a constant c
- $a_n = o(b_n)$ iff $a_n/b_n \to 0$ as $n \to \infty$

Definition 1.9

Let $X_1, X_2, ...$ be random vectors and $Y_1, Y_2, ...$ be random variables defined on a common probability space.

(i) $X_n = O(Y_n)$ a.s. iff $P(\|X_n\| = O(|Y_n|)) = 1$.

(ii) $X_n = o(Y_n)$ a.s. iff $X_n/Y_n \to_{a.s.} 0$.

(iii) $X_n = O_p(Y_n)$ iff, for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\sup_n P(\|X_n\| \ge C_\varepsilon |Y_n|) < \varepsilon.$$

(iv) $X_n = o_p(Y_n)$ iff $X_n/Y_n \to_p 0$.

Discussions and properties

- Since $a_n = O(1)$ means that $\{a_n\}$ is bounded, $\{X_n\}$ is said to be bounded in probability if $X_n = O_p(1)$.
- $X_n = o_p(Y_n)$ implies $X_n = O_p(Y_n)$
- $X_n = O_p(Y_n)$ and $Y_n = O_p(Z_n)$ implies $X_n = O_p(Z_n)$
- $X_n = O_p(Y_n)$ does not imply $Y_n = O_p(X_n)$
- If $X_n = O_p(Z_n)$, then $X_nY_n = O_p(Y_nZ_n)$.
- If $X_n = O_p(Z_n)$ and $Y_n = O_p(Z_n)$, then $X_n + Y_n = O_p(Z_n)$.
- The same conclusion can be obtained if $O_p(\cdot)$ and $o_p(\cdot)$ are replaced by $O(\cdot)$ a.s. and $o(\cdot)$ a.s., respectively.
- If $X_n \to_d X$ for a random variable X, then $X_n = O_p(1)$
- If $E|X_n| = O(a_n)$, then $X_n = O_p(a_n)$, where $a_n \in (0, \infty)$.
- If $X_n \to_{a.s.} X$, then $\sup_n |X_n| = O_p(1)$.

Relationship among convergence modes

Theorem 1.8

(i) If $X_n \to_{a.s.} X$, then $X_n \to_p X$. (The converse is not true.)

(ii) If $X_n \to_{L_r} X$ for an $r > 0$, then $X_n \to_p X$. (The converse is not true.)

(iii) If $X_n \to_p X$, then $X_n \to_d X$. (The converse is not true.)

(iv) (Skorohod's theorem). If $X_n \to_d X$, then there are random vectors $Y, Y_1, Y_2, \ldots$ defined on a common probability space such that $P_Y = P_X$, $P_{Y_n} = P_{X_n}$, $n = 1, 2, \ldots$, and $Y_n \to_{a.s.} Y$.
(A useful result; a conditional converse of (i)-(iii).)

(v) If, for every $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(\|X_n - X\| \geq \varepsilon) < \infty$, then $X_n \to_{a.s.} X$.
(A conditional converse of (i): $P(\|X_n - X\| \geq \varepsilon)$ tends to 0 fast enough.)

(vi) If $X_n \to_p X$, then there is a subsequence $\{X_{n_j}, j = 1, 2, \ldots\}$ such that $X_{n_j} \to_{a.s.} X$ as $j \to \infty$. (A partial converse of (i).)

Theorem 1.8 (continued)

(vii) If $X_n \to_d X$ and $P(X = c) = 1$, where $c \in \mathscr{R}^k$ is a constant vector, then $X_n \to_p c$. (A conditional converse of (i).)

(viii) Suppose that $X_n \to_d X$.
Then, for any $r > 0$,

$$\lim_{n\to\infty} E\|X_n\|_r^r = E\|X\|_r^r < \infty$$

[we call this moment convergence (MC)]
iff $\{\|X_n\|_r^r\}$ is *uniformly integrable* (UI) in the sense that

$$\lim_{t\to\infty} \sup_n E\left(\|X_n\|_r^r I_{\{\|X_n\|_r > t\}}\right) = 0.$$

(A conditional converse of (ii).)
In particular, $X_n \to_{L_r} X$ if and only if $\{\|X_n - X\|_r^r\}$ is UI

Discussions on uniform integrability

- If there is only one random vector, then UI is

$$\lim_{t\to\infty} E\left(\|X\|_r^r I_{\{\|X\|_r>t\}}\right) = 0,$$

  which is equivalent to the integrability of $\|X\|_r^r$ (dominated convergence theorem).

- Sufficient conditions for uniform integrability:

$$\sup_n E\|X_n\|_r^{r+\delta} < \infty \quad \text{for a } \delta > 0$$

  This is because

$$\begin{aligned}\lim_{t\to\infty}\sup_n E\left(\|X_n\|_r^r I_{\{\|X_n\|_r>t\}}\right) &\le \lim_{t\to\infty}\sup_n E\left(\|X_n\|_r^r I_{\{\|X_n\|_r>t\}}\frac{\|X_n\|_r^\delta}{t^\delta}\right)\\ &\le \lim_{t\to\infty}\frac{1}{t^\delta}\sup_n E\left(\|X_n\|_r^{r+\delta}\right)\\ &= 0\end{aligned}$$

- Exercises 117-120.

Proof of Theorem 1.8

(i) The result follows from Lemma 1.4.

(ii) The result follows from Chebyshev's inequality with $\varphi(t) = |t|^r$.

(iii) Assume $k = 1$. (The general case is proved in the textbook.)
Let x be a continuity point of F_X and $\varepsilon > 0$ be given.
Then

$$\begin{aligned} F_X(x-\varepsilon) &= P\big(X \le x-\varepsilon\big) \\ &\le P\big(X_n \le x\big) + P\big(X \le x-\varepsilon, X_n > x\big) \\ &\le F_{X_n}(x) + P\left(|X_n - X| > \varepsilon\right). \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$F_X(x-\varepsilon) \le \liminf_n F_{X_n}(x).$$

Switching X_n and X in the previous argument, we can show that

$$F_X(x+\varepsilon) \ge \limsup_n F_{X_n}(x).$$

Since ε is arbitrary and F_X is continuous at x,

$$F_X(x) = \lim_{n\to\infty} F_{X_n}(x).$$

Proof (continued)

(iv) The proof of this part can be found in Billingsley (1995, pp. 333-334).

(v) Let $A_n = \{\|X_n - X\| \geq \varepsilon\}$. The result follows from Lemma 1.4, Lemma 1.5(i), and Proposition 1.1(iii).

(vi) $X_n \to_p X$ means $\lim_{n\to\infty} P(\|X_n - X\| > \varepsilon) = 0$ for every $\varepsilon > 0$.
That is, for every $\varepsilon > 0$, $P(\|X_n - X\| > \varepsilon) < \varepsilon$ for $n > n_\varepsilon$ (n_ε is an integer depending on ε).
For every $j = 1, 2, ...$, there is a positive integer n_j such that

$$P(\|X_{n_j} - X\| > 2^{-j}) < 2^{-j}.$$

For any $\varepsilon > 0$, there is a k_ε such that for $j \geq k_\varepsilon$,
$P(\|X_{n_j} - X\| > \varepsilon) < P(\|X_{n_j} - X\| > 2^{-j})$.
Since $\sum_{j=1}^{\infty} 2^{-j} = 1$, it follows from the result in (v) that $X_{n_j} \to_{a.s.} X$ as $j \to \infty$.

(vii) The proof for this part is left as an exercise.

Properties of the quotient random variables

Proposition A1

Suppose $X, X_1, X_2, \ldots,$ are positive random variables. Then $X_n \to_{a.s.} X$ if and only if for every $\varepsilon > 0$, $\lim_{n\to\infty} P\{\sup_{k\geq n} \frac{X_k}{X} > 1+\varepsilon\} = 0$, and $\lim_{n\to\infty} P\{\sup_{k\geq n} \frac{X}{X_k} > 1+\varepsilon\} = 0$.

Proposition A2

Suppose $X, X_1, X_2, \ldots,$ are positive random variables. If $\sum_{n=1}^{\infty} P(X_n/X > 1+\varepsilon) < \infty$ and $\sum_{n=1}^{\infty} P(X/X_n > 1+\varepsilon) < \infty$, then $X_n \to_{a.s.} X$.

Homework

1. Prove these two propositions.
2. Construct two random variable sequences such that these two propositions can apply.