Definition 1.7.

Let \((\Omega, \mathcal{F}, P)\) be a probability space.

(i) Let \(\mathcal{C}\) be a collection of subsets in \(\mathcal{F}\). Events in \(\mathcal{C}\) are said to be \textit{independent} iff for any positive integer \(n\) and distinct events \(A_1, \ldots, A_n\) in \(\mathcal{C}\),

\[
P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n).
\]

(ii) Collections \(\mathcal{C}_i \subset \mathcal{F}, i \in \mathcal{I}\) (an index set that can be uncountable), are said to be independent iff events in any collection of the form \(\{A_i \in \mathcal{C}_i : i \in \mathcal{I}\}\) are independent.

(iii) Random elements \(X_i, i \in \mathcal{I}\), are said to be independent iff \(\sigma(X_i), i \in \mathcal{I}\), are independent.
Lemma 1.3 (a useful result for checking the independence of $\sigma$-fields)

Let $C_i, i \in I$, be independent collections of events. If each $C_i$ is a $\pi$-system ($A \in C_i$ and $B \in C_i$ implies $A \cap B \in C_i$), then $\sigma(C_i), i \in I$, are independent.

Facts

- Random variables $X_i, i = 1, \ldots, k$, are independent according to Definition 1.7 iff
  \[ F(x_1, \ldots, x_k)(x_1, \ldots, x_k) = F_{X_1}(x_1) \cdots F_{X_k}(x_k), \quad (x_1, \ldots, x_k) \in \mathbb{R}^k \]

  Take $C_i = \{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}\}, i = 1, \ldots, k$

- If $X$ and $Y$ are independent random vectors, then so are $g(X)$ and $h(Y)$ for Borel functions $g$ and $h$.

- Two events $A$ and $B$ are independent iff $P(B|A) = P(B)$, which means that $A$ provides no information about the probability of the occurrence of $B$.  

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- Random variables $X_i, i = 1, \ldots, k$, are independent according to Definition 1.7 iff
  
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Proposition 1.11

Let $X$ be a random variable with $E|X| < \infty$ and let $Y_i$ be random $k_i$-vectors, $i = 1, 2$.

Suppose that $(X, Y_1)$ and $Y_2$ are independent.

Then

$$E[X|(Y_1, Y_2)] = E(X|Y_1) \text{ a.s.}$$

Proof

First, $E(X|Y_1)$ is Borel on $(\Omega, \sigma(Y_1, Y_2))$, since $\sigma(Y_1) \subset \sigma(Y_1, Y_2)$.

Next, we need to show that for any Borel set $B \in B^{k_1+k_2}$,

$$\int_{(Y_1, Y_2)^{-1}(B)} XdP = \int_{(Y_1, Y_2)^{-1}(B)} E(X|Y_1)dP.$$

If $B = B_1 \times B_2$, where $B_i \in B^{k_i}$, then

$$(Y_1, Y_2)^{-1}(B) = Y_1^{-1}(B_1) \cap Y_2^{-1}(B_2)$$
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Proof (continued)

and

\[
\int_{Y_1^{-1}(B_1) \cap Y_2^{-1}(B_2)} E(X \mid Y_1) \, dP = \int I_{Y_1^{-1}(B_1)}(B_1) I_{Y_2^{-1}(B_2)}(B_2) E(X \mid Y_1) \, dP \\
= \int I_{Y_1^{-1}(B_1)}(B_1) E(X \mid Y_1) \, dP \int I_{Y_2^{-1}(B_2)}(B_2) \, dP \\
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= \int Y_1^{-1}(B_1) \cap Y_2^{-1}(B_2) \, X \, dP,
\]

where the second and the next to last equalities follow the independence of \((X, Y_1)\) and \(Y_2\), and the third equality follows from the fact that \(E(X \mid Y_1)\) is the conditional expectation of \(X\) given \(Y_1\). This shows that the result for \(B = B_1 \times B_2\).

Note that \(\mathcal{B}^{k_1} \times \mathcal{B}^{k_2}\) is a \(\pi\)-system.
Proof (continued)

We can show that the following collection is a \( \lambda \)-system:

\[ \mathcal{H} = \left\{ B \subset \mathbb{R}^{k_1+k_2} : \int_{(Y_1,Y_2)^{-1}(B)} X dP = \int_{(Y_1,Y_2)^{-1}(B)} E(X|Y_1) dP \right\} \]

Since we have already shown that \( \mathcal{B}^{k_1} \times \mathcal{B}^{k_2} \subset \mathcal{H} \), \( \mathcal{B}^{k_1+k_2} = \sigma(\mathcal{B}^{k_1} \times \mathcal{B}^{k_2}) \subset \mathcal{H} \) and thus the result follows.

Remarks

- The result in Proposition 1.11 still holds if \( X \) is replaced by \( h(X) \) for any Borel \( h \) and, hence,

  \[ P(A|Y_1, Y_2) = P(A|Y_1) \text{ a.s. for any } A \in \sigma(X), \quad (1) \]

  if \( (X, Y_1) \) and \( Y_2 \) are independent.

- We say that given \( Y_1, X \) and \( Y_2 \) are conditionally independent iff (1) holds.

- Proposition 1.11 can be stated as: if \( Y_2 \) and \( (X, Y_1) \) are independent, then given \( Y_1, X \) and \( Y_2 \) are conditionally independent.
We can show that the following collection is a $\lambda$-system:

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- We say that given $Y_1$, $X$ and $Y_2$ are conditionally independent iff (1) holds.

- Proposition 1.11 can be stated as: if $Y_2$ and $(X, Y_1)$ are independent, then given $Y_1$, $X$ and $Y_2$ are conditionally independent.
Conditional distribution

For random vectors $X$ and $Y$, is $P[X^{-1}(B)|Y = y]$ a probability measure for given $y$?

Problem: $P[X^{-1}(B)|Y = y]$ is defined a.s.

Theorem 1.7(i) (Existence of conditional distributions)

Let $X$ be a random $n$-vector on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{A}$ be a sub-$\sigma$-field of $\mathcal{F}$.

Then there exists a function $P(B, \omega)$ on $\mathcal{B}^n \times \Omega$ such that

(a) $P(B, \omega) = P[X^{-1}(B)|\mathcal{A}]$ a.s. for any fixed $B \in \mathcal{B}^n$, and

(b) $P(\cdot, \omega)$ is a probability measure on $(\mathcal{B}^n, \mathcal{B}^n)$ for any fixed $\omega \in \Omega$.

Let $Y$ be measurable from $(\Omega, \mathcal{F}, P)$ to $(\Lambda, \mathcal{G})$.

Then there exists $P_{X|Y}(B|y)$ such that

(a) $P_{X|Y}(B|y) = P[X^{-1}(B)|Y = y]$ a.s. $P_Y$ for any fixed $B \in \mathcal{B}^n$, and

(b) $P_{X|Y}(\cdot|y)$ is a probability measure on $(\mathcal{B}^n, \mathcal{B}^n)$ for any fixed $y \in \Lambda$.

Furthermore, if $E|g(X, Y)| < \infty$ with a Borel function $g$, then

$$E[g(X, Y)|Y = y] = E[g(X, y)|Y = y] = \int_{\mathcal{B}^n} g(x, y) dP_{X|Y}(x|y) \text{ a.s. } P_Y.$$
Conditional distribution

For random vectors \( X \) and \( Y \), is \( P[X^{-1}(B)|Y = y] \) a probability measure for given \( y \)?

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**Theorem 1.7(i) (Existence of conditional distributions)**

Let \( X \) be a random \( n \)-vector on a probability space \( (\Omega, \mathcal{F}, P) \) and \( \mathcal{A} \) be a sub-\( \sigma \)-field of \( \mathcal{F} \).

Then there exists a function \( P(B, \omega) \) on \( \mathcal{B}^n \times \Omega \) such that

(a) \( P(B, \omega) = P[X^{-1}(B)|\mathcal{A}] \) a.s. for any fixed \( B \in \mathcal{B}^n \), and

(b) \( P(\cdot, \omega) \) is a probability measure on \( (\mathbb{R}^n, \mathcal{B}^n) \) for any fixed \( \omega \in \Omega \).

Let \( Y \) be measurable from \( (\Omega, \mathcal{F}, P) \) to \( (\Lambda, \mathcal{G}) \).

Then there exists \( P_{X|Y}(B|y) \) such that

(a) \( P_{X|Y}(B|y) = P[X^{-1}(B)|Y = y] \) a.s. \( P_Y \) for any fixed \( B \in \mathcal{B}^n \), and

(b) \( P_{X|Y}(\cdot|y) \) is a probability measure on \( (\mathbb{R}^n, \mathcal{B}^n) \) for any fixed \( y \in \Lambda \).

Furthermore, if \( E|g(X, Y)| < \infty \) with a Borel function \( g \), then

\[
E[g(X, Y)|Y = y] = E[g(X, y)|Y = y] = \int_{\mathbb{R}^n} g(x, y) dP_{X|Y}(x|y) \text{ a.s. } P_Y.
\]
Theorem 1.7(ii)

Let $(\Lambda, \mathcal{G}, P_1)$ be a probability space. Suppose that $P_2$ is a function from $\mathcal{B}^n \times \Lambda$ to $\mathbb{R}$ and satisfies

(a) $P_2(\cdot, y)$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}^n)$ for any $y \in \Lambda$, and

(b) $P_2(B, \cdot)$ is Borel for any $B \in \mathcal{B}^n$.

Then there is a unique probability measure $P$ on $(\mathbb{R}^n \times \Lambda, \sigma(\mathcal{B}^n \times \mathcal{G}))$ such that, for $B \in \mathcal{B}^n$ and $C \in \mathcal{G}$,

$$P(B \times C) = \int_C P_2(B, y) dP_1(y). \tag{2}$$

Furthermore, if $(\Lambda, \mathcal{G}) = (\mathbb{R}^m, \mathcal{B}^m)$, and $X(x, y) = x$ and $Y(x, y) = y$ define the coordinate random vectors, then $P_Y = P_1$, $P_{X|Y}(\cdot | y) = P_2(\cdot, y)$, and the probability measure in (2) is the joint distribution of $(X, Y)$, which has the following joint c.d.f.:

$$F(x, y) = \int_{(-\infty, y]} P_{X|Y}(([-\infty, x]|z) dP_Y(z), \quad x \in \mathbb{R}^n, y \in \mathbb{R}^m, \tag{3}$$

where $(-\infty, a]$ denotes $(-\infty, a_1] \times \cdots \times (-\infty, a_k]$ for $a = (a_1, \ldots, a_k)$. 

Conditional distribution

For a fixed \( y \), \( P_{X|Y=y} = P_{X|Y}(\cdot|y) \) is called the conditional distribution of \( X \) given \( Y = y \).

Two-stage experiment theorem

If \( Y \in \mathbb{R}^m \) is selected in stage 1 of an experiment according to its marginal distribution \( P_Y = P_1 \), and \( X \) is chosen afterward according to a distribution \( P_2(\cdot, y) \), then the combined two-stage experiment produces a jointly distributed pair \( (X, Y) \) with distribution \( P_{(X,Y)} \) given by (2) and \( P_{X|Y=y} = P_2(\cdot, y) \).

This provides a way of generating dependent random variables.

Example 1.23

A market survey is conducted to study whether a new product is preferred over the product currently available in the market (old product).
**Conditional distribution**

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Example 1.23
A market survey is conducted to study whether a new product is preferred over the product currently available in the market (old product).
Example 1.23 (continued)

The survey is conducted by mail. Questionnaires are sent along with the sample products (both new and old) to \( N \) customers randomly selected from a population, where \( N \) is a positive integer. Each customer is asked to fill out the questionnaire and return it. Responses from customers are either 1 (new is better than old) or 0 (otherwise).

Some customers, however, do not return the questionnaires. Let \( X \) be the number of ones in the returned questionnaires. What is the distribution of \( X \)?

If every customer returns the questionnaire, then (from elementary probability) \( X \) has the binomial distribution \( Bi(p, N) \) in Table 1.1 (assuming that the population is large enough so that customers respond independently), where \( p \in (0, 1) \) is the overall rate of customers who prefer the new product.
Example 1.23 (continued)

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Example 1.23 (continued)

Now, let $Y$ be the number of customers who respond. Then $Y$ is random.

Suppose that customers respond independently with the same probability $\pi \in (0, 1)$.

Then $P_Y$ is the binomial distribution $Bi(\pi, N)$.

Given $Y = y$ (an integer between 0 and $N$), $P_{X|Y=y}$ is the binomial distribution $Bi(p, y)$ if $y \geq 1$ and the point mass at 0 if $y = 0$.

Using (3) and the fact that binomial distributions have p.d.f.'s w.r.t. counting measure, we obtain that the joint c.d.f. of $(X, Y)$ is

$$F(x, y) = \sum_{k=0}^{y} P_{X|Y=k}((-\infty, x]) \left(\begin{array}{c} N \\ k \end{array}\right) \pi^k (1 - \pi)^{N-k}$$

$$= \sum_{k=0}^{y} \min\{x,k\} \sum_{j=0}^{\min\{x,k\}} \binom{k}{j} p^j (1-p)^{k-j} \left(\begin{array}{c} N \\ k \end{array}\right) \pi^k (1 - \pi)^{N-k}$$

for $x = 0, 1, ..., y$, $y = 0, 1, ..., N$.

The marginal c.d.f. $F_X(x) = F(x, \infty) = F(x, N)$. 
Example 1.23 (continued)

The p.d.f. of \( X \) w.r.t. counting measure is

\[
f_X(x) = \sum_{k=x}^{N} \binom{k}{x} p^x (1 - p)^{k-x} \binom{N}{k} \pi^k (1 - \pi)^{N-k}
\]

\[
= \binom{N}{x} (\pi p)^x (1 - \pi p)^{N-x} \sum_{k=x}^{N} \binom{N-x}{k-x} \left( \frac{\pi - \pi p}{1 - \pi p} \right)^{k-x} \left( \frac{1 - \pi}{1 - \pi p} \right)^{N-k}
\]

\[
= \binom{N}{x} (\pi p)^x (1 - \pi p)^{N-x}
\]

for \( x = 0, 1, \ldots, N \).

It turns out that the marginal distribution of \( X \) is the binomial distribution \( Bi(\pi p, N) \).