Theorem 1.8

(i) If \( X_n \to_{a.s.} X \), then \( X_n \to_p X \). (The converse is not true.)

(ii) If \( X_n \to_{L_r} X \) for an \( r > 0 \), then \( X_n \to_p X \). (The converse is not true.)

(iii) If \( X_n \to_p X \), then \( X_n \to_d X \). (The converse is not true.)

(iv) (Skorohod’s theorem). If \( X_n \to_d X \), then there are random vectors \( Y, Y_1, Y_2, \ldots \) defined on a common probability space such that \( P_Y = P_X \), \( P_{Y_n} = P_{X_n} \), \( n = 1, 2, \ldots \), and \( Y_n \to_{a.s.} Y \).

(A useful result; a conditional converse of (i)-(iii).)

(v) If, for every \( \varepsilon > 0 \), \( \sum_{n=1}^{\infty} P(\|X_n - X\| \geq \varepsilon) < \infty \), then \( X_n \to_{a.s.} X \).

(A conditional converse of (i): \( P(\|X_n - X\| \geq \varepsilon) \) tends to 0 fast enough.)

(vi) If \( X_n \to_p X \), then there is a subsequence \( \{X_{n_j}, j = 1, 2, \ldots\} \) such that \( X_{n_j} \to_{a.s.} X \) as \( j \to \infty \). (A partial converse of (i).)
Theorem 1.8 (continued)

(vii) If $X_n \rightarrow_d X$ and $P(X = c) = 1$, where $c \in \mathbb{R}^k$ is a constant vector, then $X_n \rightarrow_p c$. (A conditional converse of (i).)

(viii) Suppose that $X_n \rightarrow_d X$.
Then, for any $r > 0$,

$$\lim_{n \to \infty} E \|X_n\|_r^r = E \|X\|_r^r < \infty$$

[we call this moment convergence (MC)]
iff $\{\|X_n\|_r^r\}$ is uniformly integrable (UI) in the sense that

$$\lim_{t \to \infty} \sup_n E \left(\|X_n\|_r^r I_{\{\|X_n\|_r > t\}}\right) = 0.$$  

(A conditional converse of (ii).)
In particular, $X_n \rightarrow_{L_r} X$ if and only if $\{\|X_n - X\|_r^r\}$ is UI.
Discussions on uniform integrability

- If there is only one random vector, then UI is

\[ \lim_{t \to \infty} E \left( \| X \|_r^r I\{\| X \|_r > t \} \right) = 0, \]

which is equivalent to the integrability of \( \| X \|_r^r \) (dominated convergence theorem).

- Sufficient conditions for uniform integrability:

\[ \sup_n E\| X_n \|_r^{r+\delta} < \infty \quad \text{for a } \delta > 0 \]

This is because

\[
\limsup_{t \to \infty} \sup_n E \left( \| X_n \|_r^r I\{\| X_n \|_r > t \} \right) \leq \limsup_{t \to \infty} E \left( \| X_n \|_r^r I\{\| X_n \|_r > t \} \frac{\| X_n \|_r^\delta}{t^\delta} \right) \\
\leq \lim_{t \to \infty} \frac{1}{t^\delta} \sup_n E \left( \| X_n \|_r^{r+\delta} \right) = 0
\]

- Exercises 117-120.
Proof of Theorem 1.8

(i) The result follows from Lemma 1.4.

(ii) The result follows from Chebyshev’s inequality with $\varphi(t) = |t|^r$.

(iii) Assume $k = 1$. (The general case is proved in the textbook.)

Let $x$ be a continuity point of $F_X$ and $\varepsilon > 0$ be given. Then

\[ F_X(x - \varepsilon) = P(X \leq x - \varepsilon) \]
\[ \leq P(X_n \leq x) + P(X \leq x - \varepsilon, X_n > x) \]
\[ \leq F_{X_n}(x) + P(|X_n - X| > \varepsilon). \]

Letting $n \to \infty$, we obtain that

\[ F_X(x - \varepsilon) \leq \lim \inf_n F_{X_n}(x). \]

Switching $X_n$ and $X$ in the previous argument, we can show that

\[ F_X(x + \varepsilon) \geq \lim \sup_n F_{X_n}(x). \]

Since $\varepsilon$ is arbitrary and $F_X$ is continuous at $x$,

\[ F_X(x) = \lim_{n \to \infty} F_{X_n}(x). \]
(iv) The proof of this part can be found in Billingsley (1995, pp. 333-334).

(v) Let $A_n = \{\|X_n - X\| \geq \varepsilon\}$. The result follows from Lemma 1.4, Lemma 1.5(i), and Proposition 1.1(iii).

(vi) $X_n \to_p X$ means $\lim_{n \to \infty} P(\|X_n - X\| > \varepsilon) = 0$ for every $\varepsilon > 0$. That is, for every $\varepsilon > 0$, $P(\|X_n - X\| > \varepsilon) < \varepsilon$ for $n > n_\varepsilon$ ($n_\varepsilon$ is an integer depending on $\varepsilon$).

For every $j = 1, 2, \ldots$, there is a positive integer $n_j$ such that

$$P(\|X_{n_j} - X\| > 2^{-j}) < 2^{-j}.$$ 

For any $\varepsilon > 0$, there is a $k_\varepsilon$ such that for $j \geq k_\varepsilon$,

$$P(\|X_{n_j} - X\| > \varepsilon) < P(\|X_{n_j} - X\| > 2^{-j}).$$

Since $\sum_{j=1}^\infty 2^{-j} = 1$, it follows from the result in (v) that $X_{n_j} \to_{a.s.} X$ as $j \to \infty$.

(vii) The proof for this part is left as an exercise.
Proof (continued)

First, by part (iv), we may assume that $X_n \rightarrow a.s. X$ (why?). Next, for simplicity, we consider $r = 1$ and $k = 1$ only (the general case is shown in the textbook)

UI: $\lim_{t \to \infty} \sup_n E \left( |X_n| I_{\{|X_n| > t\}} \right) = 0$

MC: $\lim_{n \to \infty} E|X_n| = E|X| < \infty$

Proof of UI implies MC

By UI, for an $\varepsilon > 0$, there is a finite $t > 0$ such that

$$\sup_n E \left( |X_n| I_{\{|X_n| > t\}} \right) < \varepsilon$$

Then

$$\sup_n E|X_n| \leq \sup_n E \left( |X_n| I_{\{|X_n| > t\}} \right) + \sup_n E \left( |X_n| I_{\{|X_n| \leq t\}} \right) < \varepsilon + t < \infty$$

By Fatou’s lemma (Theorem 1.1(i)),

$$E|X| \leq \lim inf_n E|X_n| < \sup_n E|X_n| < \infty$$
First, by part (iv), we may assume that $X_n \rightarrow_{a.s.} X$ (why?). Next, for simplicity, we consider $r = 1$ and $k = 1$ only (the general case is shown in the textbook).

**UI:** \( \lim_{t \to \infty} \sup_n E \left( |X_n| I_{\{|X_n| > t\}} \right) = 0 \)

**MC:** \( \lim_{n \to \infty} E|X_n| = E|X| < \infty \)

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**Proof of UI implies MC**

By UI, for an \( \varepsilon > 0 \), there is a finite \( t > 0 \) such that

\[
\sup_n E \left( |X_n| I_{\{|X_n| > t\}} \right) < \varepsilon
\]

Then

\[
\sup_n E|X_n| \leq \sup_n E \left( |X_n| I_{\{|X_n| > t\}} \right) + \sup_n E \left( |X_n| I_{\{|X_n| \leq t\}} \right) < \varepsilon + t < \infty
\]

By Fatou’s lemma (Theorem 1.1(i)),

\[
E|X| \leq \liminf_n E|X_n| < \sup_n E|X_n| < \infty
\]
Proof of UI implies MC

Hence, MC follows if we can show that

$$\limsup \limits_n E |X_n| \leq E |X|.$$  

For any \( \varepsilon > 0 \) and \( t > 0 \), let \( A_n = \{|X_n - X| \leq \varepsilon\} \) and \( B_n = \{|X_n| > t\} \). Then

$$E |X_n| = E(|X_n| I_{A_n^c \cap B_n}) + E(|X_n| I_{A_n^c \cap B_n^c}) + E(|X_n| I_{A_n})$$

$$\leq E(|X_n| I_{B_n}) + t P(A_n^c) + E|X_n I_{A_n}|.$$  

Since \( |X_n I_{A_n}| \leq (|X_n - X| + |X|) I_{A_n} \),

$$E |X_n I_{A_n}| \leq E[(|X_n - X| + |X|) I_{A_n}] \leq \varepsilon + E |X|.$$  

Since \( \varepsilon \) is arbitrary, \( \limsup \limits_n E |X_n I_{A_n}| \leq E |X| \).

This result and previous inequality imply that

$$\limsup \limits_n E |X_n| \leq \limsup \limits_n E(|X_n| I_{B_n}) + t \lim \limits_{n \to \infty} P(A_n^c) + E |X|.$$  

Since \( \lim \limits_{n \to \infty} P(A_n^c) = 0 \) and \( \{|X_n|\} \) is uniformly integrable, letting \( t \to \infty \) we obtain the result.
Proof of MC implies UI

Let $\xi_n = |X_n| I_{B_n^c} - |X| I_{B_n^c}$, $B_n = \{|X_n| > t\}$.

Then $\xi_n \to a.s. 0$ and $|\xi_n| \leq t + |X|$, which is integrable.

By the dominated convergence theorem, $E\xi_n \to 0$; this and UI imply

$$E(|X_n| I_{B_n}) - E(|X| I_{B_n}) \to 0.$$ 

Since $E|X| < \infty$, by the dominated convergence theorem,

$$\lim_{n \to \infty} E(|X| I_{\{|X_n-X|>t/2\}}) = 0.$$ 

From the definition of $B_n$, 

$$|X| I_{B_n} \leq |X| I_{\{|X_n-X|>t/2\}} + |X| I_{\{|X|>t/2\}}.$$ 

Hence

$$\limsup_n E(|X_n| I_{B_n}) \leq \limsup_n E(|X| I_{B_n}) \leq E(|X| I_{\{|X|>t/2\}}).$$ 

Letting $t \to \infty$, it follows from the dominated convergence theorem that

$$\lim_{t \to \infty} \limsup_n E(|X_n| I_{B_n}) \leq \lim_{t \to \infty} E(|X| I_{\{|X|>t/2\}}) = 0.$$ 

This proves UI.
Example 1.27.

As an application of Theorem 1.8(viii) and Proposition 1.15, we consider again the prediction problem in Example 1.22.

Suppose that we predict a random variable $X$ by a random $n$-vector $Y = (Y_1, \ldots, Y_n)$, all random variables are defined on $(\Omega, \mathcal{F})$.

It is shown in Example 1.22 that $X_n = E(X|Y_1, \ldots, Y_n)$ is the best predictor in terms of the mean squared prediction error, when $EX^2 < \infty$.

We now show that $X_n \to_{a.s.} X$ when $n \to \infty$ under the assumption that $\mathcal{F} = \sigma(Y_1, Y_2, \ldots)$ (i.e., $Y_1, Y_2, \ldots$ provide all information).

From the discussion in §1.4.4, $\{X_n\}$ is a martingale.

Also, $\sup_n E|X_n| \leq \sup_n E[E(|X||Y_1, \ldots, Y_n)] = E|X| < \infty$.

Hence, by Proposition 1.15, $X_n \to_{a.s.} Z$ for some random variable $Z$.

We now need to show $Z = X$ a.s.

Since $EX_n^2 \leq EX^2 < \infty$ (why?), $\{|X_n|\}$ is uniformly integrable (why?).
Example 1.27 (continued)

By Theorem 1.8(viii), $E|X_n - Z| \to 0$, which implies $\int_A X_n dP \to \int_A Z dP$ for any event $A$.

Note that if $A \in \sigma(Y_1, \ldots, Y_n)$, then $\int_A X_n dP = \int_A X dP$.

Also, $\sigma(Y_1, \ldots, Y_n) \subset \sigma(Y_1, \ldots, Y_m)$ if $m > n$

Therefore, for any $A \in \cup_{j=1}^\infty \sigma(Y_1, \ldots, Y_j)$, $\int_A X dP = \int_A Z dP$.

Since $\cup_{j=1}^\infty \sigma(Y_1, \ldots, Y_j)$ generates $\sigma(Y_1, Y_2, \ldots) = \mathcal{F}$, we conclude that $\int_A X dP = \int_A Z dP$ for any $A \in \mathcal{F}$ and thus $Z = X$ a.s.

In the proof above, the condition $EX^2 < \infty$ is used only for showing the uniform integrability of $\{|X_n|\}$.

But by Exercise 120, $\{|X_n|\}$ is uniformly integrable as long as $E|X| < \infty$.

Hence $X_n \to a.s. X$ is still true if the condition $EX^2 < \infty$ is replaced by $E|X| < \infty$. 