Lecture 12: Weak convergence

Tightness

A sequence \( \{P_n\} \) of probability measures on \((\mathbb{R}^k, \mathcal{B}^k)\) is **tight** if for every \( \varepsilon > 0 \), there is a compact set \( C \subset \mathbb{R}^k \) such that \( \inf_n P_n(C) > 1 - \varepsilon \).

**Proposition 1.17**

Let \( \{P_n\} \) be a sequence of probability measures on \((\mathbb{R}^k, \mathcal{B}^k)\).

(i) Tightness of \( \{P_n\} \) is a necessary and sufficient condition that for every subsequence \( \{P_{n_i}\} \) there exists a further subsequence \( \{P_{n_j}\} \subset \{P_{n_i}\} \) and a probability measure \( P \) on \((\mathbb{R}^k, \mathcal{B}^k)\) such that \( P_{n_j} \xrightarrow{w} P \) as \( j \to \infty \).

(ii) If \( \{P_n\} \) is tight and if each subsequence that converges weakly at all converges to the same probability measure \( P \), then \( P_n \xrightarrow{w} P \).

**Proof**

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A sequence \( \{P_n\} \) of probability measures on \((\mathbb{R}^k, \mathcal{B}^k)\) is tight if for every \( \varepsilon > 0 \), there is a compact set \( C \subset \mathbb{R}^k \) such that \( \inf_n P_n(C) > 1 - \varepsilon \).

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A sequence \( \{P_n\} \) of probability measures on \((\mathbb{R}^k, \mathcal{B}^k)\) is **tight** if for every \( \varepsilon > 0 \), there is a compact set \( C \subset \mathbb{R}^k \) such that \( \inf_n P_n(C) > 1 - \varepsilon \).

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(ii) If \( \{P_n\} \) is tight and if each subsequence that converges weakly at all converges to the same probability measure \( P \), then \( P_n \to_w P \).

**Proof**

Remark

If \( \{X_n\} \) is a sequence of random \( k \)-vectors, then the tightness of \( \{P_{X_n}\} \) is the same as the boundedness of \( \{\|X_n\|\} \) in probability \( (\|X_n\| = O_p(1)) \), i.e., for any \( \varepsilon > 0 \), there is a constant \( C_\varepsilon > 0 \) such that \( \sup_n P(\|X_n\| \geq C_\varepsilon) < \varepsilon \).

Theorem 1.9 (useful sufficient and necessary conditions for convergence in distribution)

Let \( X, X_1, X_2, \ldots \) be random \( k \)-vectors.

(i) \( X_n \xrightarrow{d} X \) is equivalent to any one of the following conditions:

(a) \( E[h(X_n)] \to E[h(X)] \) for every bounded continuous function \( h \);
(b) \( \limsup_n P_{X_n}(C) \leq P_X(C) \) for any closed set \( C \subset \mathbb{R}^k \);
(c) \( \liminf_n P_{X_n}(O) \geq P_X(O) \) for any open set \( O \subset \mathbb{R}^k \).

(ii) (Lévy-Cramér continuity theorem). Let \( \phi_X, \phi_{X_1}, \phi_{X_2}, \ldots \) be the ch.f.'s of \( X, X_1, X_2, \ldots \), respectively.

\( X_n \xrightarrow{d} X \) iff \( \lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t) \) for all \( t \in \mathbb{R}^k \).

(iii) (Cramér-Wold device). \( X_n \xrightarrow{d} X \) iff \( c^\top X_n \xrightarrow{d} c^\top X \) for every \( c \in \mathbb{R}^k \).
Remark
If \( \{X_n\} \) is a sequence of random \( k \)-vectors, then the tightness of \( \{P_{X_n}\} \) is the same as the boundedness of \( \{\|X_n\|\} \) in probability \((\|X_n\| = O_p(1))\), i.e., for any \( \varepsilon > 0 \), there is a constant \( C_\varepsilon > 0 \) such that \( \sup_n P(\|X_n\| \geq C_\varepsilon) < \varepsilon \).

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   (a) \( E[h(X_n)] \to E[h(X)] \) for every bounded continuous function \( h \);
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    \( X_n \xrightarrow{d} X \) iff \( \lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t) \) for all \( t \in \mathbb{R}^k \).

(iii) (Cramér-Wold device). \( X_n \xrightarrow{d} X \) iff \( c^\tau X_n \xrightarrow{d} c^\tau X \) for every \( c \in \mathbb{R}^k \).
Proof of Theorem 1.9(i)

First, we show $X_n \rightarrow_d X$ implies (a).

By Theorem 1.8(iv) (Skorohod’s theorem), there exists a sequence of random vectors $\{Y_n\}$ and a random vector $Y$ such that $P_{Y_n} = P_{X_n}$ for all $n$, $P_Y = P_X$ and $Y_n \rightarrow_{a.s.} Y$.

For bounded continuous $h$, $h(Y_n) \rightarrow_{a.s.} h(Y)$ and, by the dominated convergence theorem, $E[h(Y_n)] \rightarrow E[h(Y)]$.

(a) follows from $E[h(X_n)] = E[h(Y_n)]$ for all $n$ and $E[h(X)] = E[h(Y)]$.

Next, we show (a) implies (b).

Let $C$ be a closed set and $f_C(x) = \inf\{\|x - y\| : y \in C\}$.

Then $f_C$ is continuous.

For $j = 1, 2, \ldots$, define $\varphi_j(t) = I_{(-\infty, 0]} + (1 - jt)I_{(0, j^{-1}]}$.

Then $h_j(x) = \varphi_j(f_C(x))$ is continuous and bounded, $h_j \geq h_{j+1}$, $j = 1, 2, \ldots$, and $h_j(x) \rightarrow l_C(x)$ as $j \rightarrow \infty$.

Hence $\limsup_n P_{X_n}(C) \leq \lim_{n \rightarrow \infty} E[h_j(X_n)] = E[h_j(X)]$ for each $j$ (by (a)).

By the dominated convergence theorem,

$E[h_j(X)] \rightarrow E[l_C(X)] = P_X(C)$.

This proves (b).
Proof of Theorem 1.9(i)

First, we show \( X_n \rightarrow_d X \) implies (a).

By Theorem 1.8(iv) (Skorohod’s theorem), there exists a sequence of random vectors \( \{ Y_n \} \) and a random vector \( Y \) such that \( P_{Y_n} = P_{X_n} \) for all \( n \), \( P_Y = P_X \) and \( Y_n \rightarrow_a.s. Y \).

For bounded continuous \( h \), \( h(Y_n) \rightarrow_a.s. h(Y) \) and, by the dominated convergence theorem, \( E[h(Y_n)] \rightarrow E[h(Y)] \).

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Then \( h_j(x) = \varphi_j(f_C(x)) \) is continuous and bounded, \( h_j \geq h_{j+1} \), \( j = 1, 2, \ldots \), and \( h_j(x) \rightarrow I_C(x) \) as \( j \rightarrow \infty \).

Hence \( \limsup_n P_{X_n}(C) \leq \lim_{n \rightarrow \infty} E[h_j(X_n)] = E[h_j(X)] \) for each \( j \) (by (a)).

By the dominated convergence theorem, \( E[h_j(X)] \rightarrow E[I_C(X)] = P_X(C) \).

This proves (b).
Proof of Theorem 1.9(i) (continued)

For any open set $O$, $O^c$ is closed. Hence, (b) is equivalent to (c).

To complete the proof we now show that (b) and (c) imply $X_n \xrightarrow{d} X$. For $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, let $(-\infty, x] = (-\infty, x_1] \times \cdots \times (-\infty, x_k]$ and $(-\infty, x) = (-\infty, x_1) \times \cdots \times (-\infty, x_k)$.

From (b) and (c),

$$P_X((-\infty, x)) \leq \liminf_n P_{X_n}((-\infty, x)) \leq \liminf_n F_{X_n}(x)$$

$$\leq \limsup_n F_{X_n}(x) = \limsup_n P_{X_n}((-\infty, x]) \leq P_X((-\infty, x]) = F_X(x).$$

If $x$ is a continuity point of $F_X$, then $P_X((-\infty, x)) = F_X(x)$.

This proves $X_n \xrightarrow{d} X$.

Proof of Theorem 1.9(ii)

From (a) of part (i), $X_n \xrightarrow{d} X$ implies $\phi_{X_n}(t) \xrightarrow{} \phi_X(t)$, since $\mathrm{e}^{\sqrt{-1} t^{\tau} x} = \cos(t^{\tau} x) + \sqrt{-1} \sin(t^{\tau} x)$ and $\cos(t^{\tau} x)$ and $\sin(t^{\tau} x)$ are bounded continuous functions for any fixed $t$. 
Proof of Theorem 1.9(i) (continued)

For any open set $O$, $O^c$ is closed.
Hence, (b) is equivalent to (c).

To complete the proof we now show that (b) and (c) imply $X_n \to_d X$.

For $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, let $(-\infty, x] = (-\infty, x_1] \times \cdots \times (-\infty, x_k]$ and $(-\infty, x) = (-\infty, x_1) \times \cdots \times (-\infty, x_k)$.

From (b) and (c),

$$P_X((-\infty, x)) \leq \liminf_n P_{X_n}((-\infty, x)) \leq \liminf_n F_{X_n}(x)$$

$$\leq \limsup_n F_{X_n}(x) = \limsup_n P_{X_n}((-\infty, x]) \leq P_X((-\infty, x]) = F_X(x).$$

If $x$ is a continuity point of $F_X$, then $P_X((-\infty, x)) = F_X(x)$.
This proves $X_n \to_d X$.

Proof of Theorem 1.9(ii)

From (a) of part (i), $X_n \to_d X$ implies $\phi_{X_n}(t) \to \phi_X(t)$, since

$$e^{\sqrt{-1}t^\tau x} = \cos(t^\tau x) + \sqrt{-1}\sin(t^\tau x)$$

and $\cos(t^\tau x)$ and $\sin(t^\tau x)$ are bounded continuous functions for any fixed $t$. 
Proof of Theorem 1.9(ii) (continued)

Suppose that \( k = 1 \) and that \( \phi_X(n)(t) \rightarrow \phi_X(t) \) for every \( t \in \mathbb{R} \).
We want to show that \( X_n \rightarrow_d X \).
We first show that \( \{P_{X_n}\} \) is tight.

By Fubini’s theorem,

\[
\frac{1}{u} \int_{-u}^{u} \left[ 1 - \phi_{X_n}(t) \right] dt = \int_{-\infty}^{\infty} \left[ \frac{1}{u} \int_{-u}^{u} \left( 1 - e^{\sqrt{-1}tx} \right) dt \right] dP_{X_n}(x)
\]

\[
= 2 \int_{-\infty}^{\infty} \left( 1 - \frac{\sin ux}{ux} \right) dP_{X_n}(x)
\]

\[
\geq 2 \int_{\{|x|>2u^{-1}\}} \left( 1 - \frac{1}{|ux|} \right) dP_{X_n}(x)
\]

\[
\geq P_{X_n} \left( (-\infty, -2u^{-1}) \cup (2u^{-1}, \infty) \right)
\]

for any \( u > 0 \).

Since \( \phi_X \) is continuous at 0 and \( \phi_X(0) = 1 \), for any \( \varepsilon > 0 \) there is a \( u > 0 \) such that \( u^{-1} \int_{-u}^{u} [1 - \phi_X(t)] dt < \varepsilon / 2 \).
Proof of Theorem 1.9(ii) (continued)

Since \( \phi_{X_n} \to \phi_X \), by the dominated convergence theorem,

\[
\sup_n \left\{ u^{-1} \int_{-u}^{u} [1 - \phi_{X_n}(t)] dt \right\} < \varepsilon.
\]

Hence,

\[
\inf_n P_{X_n} \left( [-2u^{-1}, 2u^{-1}] \right) \geq 1 - \sup_n \left\{ \frac{1}{u} \int_{-u}^{u} [1 - \phi_{X_n}(t)] dt \right\} \geq 1 - \varepsilon,
\]

i.e., \( \{P_{X_n}\} \) is tight.

Let \( \{P_{X_{n_j}}\} \) be any subsequence that converges to a probability measure \( P \).

By the first part of the proof, \( \phi_{X_{n_j}} \to \phi \), which is the ch.f. of \( P \).

By the convergence of \( \phi_{X_n} \), \( \phi = \phi_X \).

By the uniqueness theorem, \( P = P_X \).

By Proposition 1.17(ii), \( X_n \to_d X \).
Proof of Theorem 1.9(ii) (continued)

Consider now the case where $k \geq 2$ and $\phi_{X_n} \rightarrow \phi_X$.
Let $Y_{nj}$ be the $j$th component of $X_n$ and $Y_j$ be the $j$th component of $X$.
Then $\phi_{Y_{nj}} \rightarrow \phi_{Y_j}$ for each $j$.
By the proof for the case of $k = 1$, $Y_{nj} \rightarrow_d Y_j$.

By Proposition 1.17(i), $\{P_{Y_{nj}}\}$ is tight, $j = 1, \ldots, k$.
This implies that $\{P_{X_n}\}$ is tight (why?).
Then the proof for $X_n \rightarrow_d X$ is the same as that for the case of $k = 1$.

Proof of Theorem 1.9(iii)

Note that $\phi_{c^\tau X_n}(u) = \phi_{X_n}(uc)$ and $\phi_{c^\tau X}(u) = \phi_X(uc)$ for any $u \in \mathbb{R}$ and any $c \in \mathbb{R}^k$.
Hence, convergence of $\phi_{X_n}$ to $\phi_X$ is equivalent to convergence of $\phi_{c^\tau X_n}$ to $\phi_{c^\tau X}$ for every $c \in \mathbb{R}^k$.
Then the result follows from part (ii).
Consider now the case where $k \geq 2$ and $\phi_{X_n} \rightarrow \phi_X$. Let $Y_{n_j}$ be the $j$th component of $X_n$ and $Y_j$ be the $j$th component of $X$. Then $\phi_{Y_{n_j}} \rightarrow \phi_{Y_j}$ for each $j$.

By the proof for the case of $k = 1$, $Y_{n_j} \rightarrow_d Y_j$.

By Proposition 1.17(i), $\{P_{Y_{n_j}}\}$ is tight, $j = 1, ..., k$.

This implies that $\{P_{X_n}\}$ is tight (why?). Then the proof for $X_n \rightarrow_d X$ is the same as that for the case of $k = 1$.

Proof of Theorem 1.9(iii)

Note that $\phi_{c^\tau X_n}(u) = \phi_{X_n}(uc)$ and $\phi_{c^\tau X}(u) = \phi_{X}(uc)$ for any $u \in \mathbb{R}$ and any $c \in \mathbb{R}^k$.

Hence, convergence of $\phi_{X_n}$ to $\phi_X$ is equivalent to convergence of $\phi_{c^\tau X_n}$ to $\phi_{c^\tau X}$ for every $c \in \mathbb{R}^k$.

Then the result follows from part (ii).
Example 1.28

Let $X_1, \ldots, X_n$ be independent random variables having a common c.d.f. and $T_n = X_1 + \cdots + X_n$, $n = 1, 2, \ldots$.

Suppose that $E|X_1| < \infty$.

It follows from a result in calculus that the ch.f. of $X_1$ satisfies

$$\phi_{X_1}(t) = \phi_{X_1}(0) + \sqrt{-1} \mu t + o(|t|)$$

as $|t| \to 0$, where $\mu = EX_1$.

Then, the ch.f. of $T_n/n$ is

$$\phi_{T_n/n}(t) = \left[ \phi_{X_1} \left( \frac{t}{n} \right) \right]^n = \left[ 1 + \frac{\sqrt{-1} \mu t}{n} + o \left( \frac{t}{n} \right) \right]^n \to e^{\sqrt{-1} \mu t}$$

for any $t \in \mathbb{R}$ as $n \to \infty$, because $(1 + c_n/n)^n \to e^c$ for any complex sequence \{c_n\} satisfying $c_n \to c$.

$e^{\sqrt{-1} \mu t}$ is the ch.f. of the point mass probability measure at $\mu$.

By Theorem 1.9(ii), $T_n/n \to_d \mu$.

From Theorem 1.8(vii), this also shows that $T_n/n \to_p \mu$. 
Example 1.28

Let $X_1, \ldots, X_n$ be independent random variables having a common c.d.f. and $T_n = X_1 + \cdots + X_n$, $n = 1, 2, \ldots$.

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It follows from a result in calculus that the ch.f. of $X_1$ satisfies

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Then, the ch.f. of $T_n/n$ is

$$\phi_{T_n/n}(t) = \left[ \phi_{X_1}\left(\frac{t}{n}\right) \right]^n = \left[ 1 + \frac{\sqrt{-1}\mu t}{n} + o\left(\frac{t}{n}\right) \right]^n \to e^{\sqrt{-1}\mu t}$$

for any $t \in \mathbb{R}$ as $n \to \infty$, because $(1 + c_n/n)^n \to e^c$ for any complex sequence $\{c_n\}$ satisfying $c_n \to c$.

$e^{\sqrt{-1}\mu t}$ is the ch.f. of the point mass probability measure at $\mu$.

By Theorem 1.9(ii), $T_n/n \to_d \mu$.

From Theorem 1.8(vii), this also shows that $T_n/n \to_p \mu$. 
Example 1.28 (continued)

Similarly, $\mu = 0$ and $\sigma^2 = \text{var}(X_1) < \infty$ imply

$$
\phi_{T_n/\sqrt{n}}(t) = \left[1 - \frac{\sigma^2 t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to e^{-\sigma^2 t^2 / 2}
$$

for any $t \in \mathbb{R}$ as $n \to \infty$.

$e^{-\sigma^2 t^2 / 2}$ is the ch.f. of $N(0, \sigma^2)$.

Hence $T_n/\sqrt{n} \to_d N(0, \sigma^2)$.

If $\mu \neq 0$, a transformation of $Y_i = X_i - \mu$ leads to

$$(T_n - n\mu)/\sqrt{n} \to_d N(0, \sigma^2).$$

Suppose now that $X_1, \ldots, X_n$ are random $k$-vectors and $\mu = EX_1$ and $\Sigma = \text{var}(X_1)$ are finite.

For any fixed $c \in \mathbb{R}^k$, it follows from the previous discussion that

$$(c^\top T_n - nc^\top \mu)/\sqrt{n} \to_d N(0, c^\top \Sigma c).$$

From Theorem 1.9(iii) and a property of the normal distribution (Exercise 81), we conclude that

$$(T_n - n\mu)/\sqrt{n} \to_d N_k(0, \Sigma).$$
Example 1.28 (continued)

Similarly, $\mu = 0$ and $\sigma^2 = \text{var}(X_1) < \infty$ imply

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\phi_{T_n/\sqrt{n}}(t) = \left[1 - \frac{\sigma^2 t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to e^{-\sigma^2 t^2/2}
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Hence $T_n/\sqrt{n} \to_d N(0, \sigma^2)$.

If $\mu \neq 0$, a transformation of $Y_i = X_i - \mu$ leads to

$$
\frac{T_n - n\mu}{\sqrt{n}} \to_d N(0, \sigma^2).
$$

Suppose now that $X_1, \ldots, X_n$ are random $k$-vectors and $\mu = E X_1$ and $\Sigma = \text{var}(X_1)$ are finite.

For any fixed $c \in \mathbb{R}^k$, it follows from the previous discussion that

$$
\frac{c^T T_n - nc^T \mu}{\sqrt{n}} \to_d N(0, c^T \Sigma c).
$$

From Theorem 1.9(iii) and a property of the normal distribution (Exercise 81), we conclude that

$$
\frac{T_n - n\mu}{\sqrt{n}} \to_d N_k(0, \Sigma).
$$
Example 1.29

Let $X_1, \ldots, X_n$ be independent random variables having a common Lebesgue p.d.f. $f(x) = (1 - \cos x)/(\pi x^2)$.

Then the ch.f. of $X_1$ is $\max\{1 - |t|, 0\}$ (Exercise 73) and the ch.f. of $T_n/n = (X_1 + \cdots + X_n)/n$ is

$$\left(\max\left\{1 - \frac{|t|}{n}, 0\right\}\right)^n \to e^{-|t|}, \quad t \in \mathbb{R}.$$ 

Since $e^{-|t|}$ is the ch.f. of the Cauchy distribution $C(0, 1)$ (Table 1.2), we conclude that $T_n/n \to_d X$, where $X$ has the Cauchy distribution $C(0, 1)$.

- Does this result contradict the first result in Example 1.28?
- Other examples are given in Exercises 135-140.

The next result can be used to check whether $X_n \to_d X$ when $X$ has a p.d.f. $f$ and $X_n$ has a p.d.f. $f_n$. 
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Let $X_1, \ldots, X_n$ be independent random variables having a common Lebesgue p.d.f. $f(x) = (1 - \cos x)/(\pi x^2)$. Then the ch.f. of $X_1$ is $\max\{1 - |t|, 0\}$ (Exercise 73) and the ch.f. of $T_n/n = (X_1 + \cdots + X_n)/n$ is

$$\left( \max \left\{ 1 - \frac{|t|}{n}, 0 \right\} \right)^n \rightarrow e^{-|t|}, \quad t \in \mathbb{R}.$$ 

Since $e^{-|t|}$ is the ch.f. of the Cauchy distribution $C(0, 1)$ (Table 1.2), we conclude that $T_n/n \rightarrow_d X$, where $X$ has the Cauchy distribution $C(0, 1)$.

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Let $X_1, \ldots, X_n$ be independent random variables having a common Lebesgue p.d.f. $f(x) = (1 - \cos x)/(\pi x^2)$. Then the ch.f. of $X_1$ is $\max\{1 - |t|, 0\}$ (Exercise 73) and the ch.f. of $T_n/n = (X_1 + \cdots + X_n)/n$ is

$$
\left( \max\left\{1 - \frac{|t|}{n}, 0\right\} \right)^n \to e^{-|t|}, \quad t \in \mathbb{R}.
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Since $e^{-|t|}$ is the ch.f. of the Cauchy distribution $C(0, 1)$ (Table 1.2), we conclude that $T_n/n \to_d X$, where $X$ has the Cauchy distribution $C(0, 1)$.

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The next result can be used to check whether $X_n \to_d X$ when $X$ has a p.d.f. $f$ and $X_n$ has a p.d.f. $f_n$. 
Proposition 1.18 (Scheffé’s theorem)

Let \( \{f_n\} \) be a sequence of p.d.f.’s on \( \mathbb{R}^k \) w.r.t. a measure \( \nu \). Suppose that \( \lim_{n \to \infty} f_n(x) = f(x) \) a.e. \( \nu \) and \( f(x) \) is a p.d.f. w.r.t. \( \nu \). Then \( \lim_{n \to \infty} \int |f_n(x) - f(x)| \, d\nu = 0. \)

Proof

Let \( g_n(x) = [f(x) - f_n(x)] I_{\{f \geq f_n\}}(x), n = 1, 2, \ldots \)

Then

\[
\int |f_n(x) - f(x)| \, d\nu = 2 \int g_n(x) \, d\nu.
\]

Since \( 0 \leq g_n(x) \leq f(x) \) for all \( x \) and \( g_n \to 0 \) a.e. \( \nu \), the result follows from the dominated convergence theorem.

As an example, consider the Lebesgue p.d.f. \( f_n \) of the t-distribution \( t_n \) (Table 1.2), \( n = 1, 2, \ldots \).

One can show (exercise) that \( f_n \to f \), where \( f \) is the p.d.f. of \( N(0,1) \). This is an important result in statistics.
Proposition 1.18 (Scheffé’s theorem)

Let \( \{f_n\} \) be a sequence of p.d.f.’s on \( \mathbb{R}^k \) w.r.t. a measure \( \nu \).
Suppose that \( \lim_{n \to \infty} f_n(x) = f(x) \) a.e. \( \nu \) and \( f(x) \) is a p.d.f. w.r.t. \( \nu \).
Then \( \lim_{n \to \infty} \int |f_n(x) - f(x)| \, d\nu = 0. \)

Proof

Let \( g_n(x) = [f(x) - f_n(x)]I_{\{f \geq f_n\}}(x), \, n = 1, 2, \ldots \)
Then
\[
\int |f_n(x) - f(x)| \, d\nu = 2 \int g_n(x) \, d\nu.
\]
Since \( 0 \leq g_n(x) \leq f(x) \) for all \( x \) and \( g_n \to 0 \) a.e. \( \nu \), the result follows from the dominated convergence theorem.

As an example, consider the Lebesgue p.d.f. \( f_n \) of the t-distribution \( t_n \) (Table 1.2), \( n = 1, 2, \ldots \).
One can show (exercise) that \( f_n \to f \), where \( f \) is the p.d.f. of \( N(0,1) \).
This is an important result in statistics.
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