Transformation and convergence

- Transformation is an important tool in statistics.
- If $X_n$ converges to $X$ in some sense, we often need to check whether $g(X_n)$ converges to $g(X)$ in the same sense.
- The continuous mapping theorem provides an answer to the question in many problems.

Theorem 1.10. Continuous mapping theorem

Let $X, X_1, X_2, \ldots$ be random $k$-vectors defined on a probability space and $g$ be a measurable function from $(\mathbb{R}^k, \mathcal{B}^k)$ to $(\mathbb{R}^l, \mathcal{B}^l)$. Suppose that $g$ is continuous a.s. $P_X$. Then

(i) $X_n \xrightarrow{a.s.} X$ implies $g(X_n) \xrightarrow{a.s.} g(X)$;
(ii) $X_n \xrightarrow{p} X$ implies $g(X_n) \xrightarrow{p} g(X)$;
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Proof

(i) can be established using a result in calculus.

(iii) follows from Theorem 1.9(i): for any bounded and continuous \( h \), 
\[
E[h(g(X_n))] \to E[h(g(X))],
\]
since \( h \circ g \) is bounded and continuous.

We prove (ii) for the special case of \( X = c \) (a constant).
From the continuity of \( g \), for any \( \varepsilon > 0 \), there is a \( \delta_\varepsilon > 0 \) such that
\[
\|g(x) - g(c)\| < \varepsilon \quad \text{whenever} \quad \|x - c\| < \delta_\varepsilon.
\]
Hence,
\[
\{ \omega : \|g(X_n(\omega)) - g(c)\| < \varepsilon \} \supset \{ \omega : \|X_n(\omega) - c\| < \delta_\varepsilon \}
\]
and
\[
P(\|g(X_n) - g(c)\| \geq \varepsilon) \leq P(\|X_n - c\| \geq \delta_\varepsilon).
\]
Hence \( g(X_n) \to_p g(c) \) follows from \( X_n \to_p c \).

Is the previous argument still valid when \( c \) is replaced by the random vector \( X \) in the general case? If not, how do we fix the proof?
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If not, how do we fix the proof?
Example 1.30.

(i) Let $X_1, X_2, \ldots$ be random variables. If $X_n \xrightarrow{d} X$, where $X$ has the $N(0,1)$ distribution, then $X_n^2 \xrightarrow{d} Y$, where $Y$ has the chi-square distribution $\chi_1^2$.

(ii) Let $(X_n, Y_n)$ be random 2-vectors satisfying $(X_n, Y_n) \xrightarrow{d} (X, Y)$, where $X$ and $Y$ are independent random variables having the $N(0,1)$ distribution. Then $X_n/Y_n \xrightarrow{d} X/Y$, which has the Cauchy distribution $C(0,1)$.

(iii) Under the conditions in part (ii), $\max\{X_n, Y_n\} \xrightarrow{d} \max\{X, Y\}$, which has the c.d.f. $[\Phi(x)]^2$ ($\Phi(x)$ is the c.d.f. of $N(0,1)$).

In Example 1.30(ii) and (iii), the condition that $(X_n, Y_n) \xrightarrow{d} (X, Y)$ cannot be relaxed to $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ (exercise); i.e., we need the convergence of the joint c.d.f. of $(X_n, Y_n)$.

This is different when $\xrightarrow{d}$ is replaced by $\xrightarrow{p}$ or $\xrightarrow{a.s.}$.

The next result, which plays an important role in statistics, establishes the convergence in distribution of $X_n + Y_n$ or $X_n Y_n$ when no information regarding the joint c.d.f. of $(X_n, Y_n)$ is provided.
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Theorem 1.11 (Slutsky’s theorem)

Let $X, X_1, X_2, ..., Y_1, Y_2, ...$ be random variables on a probability space. Suppose that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, where $c$ is a constant. Then

(i) $X_n + Y_n \xrightarrow{d} X + c$;
(ii) $Y_n X_n \xrightarrow{d} cX$;
(iii) $X_n / Y_n \xrightarrow{d} X / c$ if $c \neq 0$.

Proof

We prove (i) only. The proofs of (ii) and (iii) are left as exercises.

Let $t \in \mathbb{R}$ and $\varepsilon > 0$ be fixed constants.

Then

$$F_{X_n + Y_n}(t) = P(X_n + Y_n \leq t)$$

$$\leq P(\{X_n + Y_n \leq t\} \cap \{|Y_n - c| < \varepsilon\}) + P(|Y_n - c| \geq \varepsilon)$$

$$\leq P(X_n \leq t - c + \varepsilon) + P(|Y_n - c| \geq \varepsilon)$$
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Proof

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$$F_{X_n+Y_n}(t) = P(X_n + Y_n \leq t) \leq P\left(\{X_n + Y_n \leq t\} \cap \{|Y_n - c| < \varepsilon\}\right) + P\left(|Y_n - c| \geq \varepsilon\right) \leq P(X_n \leq t - c + \varepsilon) + P(|Y_n - c| \geq \varepsilon)$$
Similarly,

\[ F_{X_n + Y_n}(t) \geq P(X_n \leq t - c - \varepsilon) - P(|Y_n - c| \geq \varepsilon). \]

If \( t - c, t - c + \varepsilon, \) and \( t - c - \varepsilon \) are continuity points of \( F_X \), then it follows from the previous two inequalities and the hypotheses of the theorem that

\[ F_X(t - c - \varepsilon) \leq \liminf_{n \rightarrow \infty} F_{X_n + Y_n}(t) \leq \limsup_{n \rightarrow \infty} F_{X_n + Y_n}(t) \leq F_X(t - c + \varepsilon). \]

Since \( \varepsilon \) can be arbitrary (why?),

\[ \lim_{n \rightarrow \infty} F_{X_n + Y_n}(t) = F_X(t - c). \]

The result follows from \( F_{X+c}(t) = F_X(t - c). \)
Similarly,

\[ F_{X_n + Y_n}(t) \geq P(X_n \leq t - c - \varepsilon) - P(|Y_n - c| \geq \varepsilon). \]

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The result follows from \( F_{X + c}(t) = F_X(t - c) \).

An application of Theorem 1.11 is given in the proof of the following important result.
Theorem 1.12

Let $X_1, X_2, \ldots$ and $Y = (Y_1, \ldots, Y_k)$ be random $k$-vectors satisfying

$$a_n(X_n - c) \to_d Y,$$

where $c \in \mathbb{R}^k$ and $\{a_n\}$ is a sequence of positive numbers with $\lim_{n \to \infty} a_n = \infty$.

Let $g$ be a function from $\mathbb{R}^k$ to $\mathbb{R}$.

(i) If $g$ is differentiable at $c$, then

$$a_n[g(X_n) - g(c)] \to_d [\nabla g(c)]^\top Y,$$

where $\nabla g(x)$ denotes the $k$-vector of partial derivatives of $g$ at $x$.

(ii) Suppose that $g$ has continuous partial derivatives of order $m > 1$ in a neighborhood of $c$, with all the partial derivatives of order $j$, $1 \leq j \leq m - 1$, vanishing at $c$, but with the $m$th-order partial derivatives not all vanishing at $c$.

Then

$$a_n^m[g(X_n) - g(c)] \to_d \frac{1}{m!} \sum_{i_1=1}^{k} \cdots \sum_{i_m=1}^{k} \frac{\partial^m g}{\partial x_{i_1} \cdots \partial x_{i_m}} \bigg|_{x=c} Y_{i_1} \cdots Y_{i_m}.$$
Proof

We prove (i) only.

Let

\[ Z_n = a_n[g(X_n) - g(c)] - a_n[\nabla g(c)]^\tau (X_n - c). \]

If we can show that \( Z_n = o_p(1) \), then by \( a_n(X_n - c) \to_d Y \), Theorem 1.9(iii), and Theorem 1.11(i), result (i) holds.

The differentiability of \( g \) at \( c \) implies that for any \( \varepsilon > 0 \), there is a \( \delta_\varepsilon > 0 \) such that

\[ |g(x) - g(c) - [\nabla g(c)]^\tau (x - c)| \leq \varepsilon \|x - c\| \]

whenever \( \|x - c\| < \delta_\varepsilon \).

Then for a fixed \( \eta > 0 \),

\[ P(\|Z_n\| \geq \eta) \leq P(\|X_n - c\| \geq \delta_\varepsilon) + P(a_n\|X_n - c\| \geq \eta/\varepsilon). \]

Since \( a_n \to \infty \), \( a_n(X_n - c) \to_d Y \) and Theorem 1.11(ii) imply \( X_n \to_p c \).

By Theorem 1.10(iii), \( a_n(X_n - c) \to_d Y \) implies \( a_n\|X_n - c\| \to_d \|Y\| \).

Without loss of generality, we can assume that \( \eta/\varepsilon \) is a continuity point of \( F_{\|Y\|} \).
Proof (continued)

Then

\[
\limsup_{n} P(|Z_n| \geq \eta) \leq \lim_{n \to \infty} P(\|X_n - c\| \geq \delta_\varepsilon) \\
+ \lim_{n \to \infty} P(a_n\|X_n - c\| \geq \eta/\varepsilon) \\
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\(Z_n \to_p 0\) follows since \(\varepsilon\) can be arbitrary.

Remarks

- In statistics, we often need a nondegenerated limiting distribution of \(a_n[g(X_n) - g(c)]\) so that probabilities involving \(a_n[g(X_n) - g(c)]\) can be approximated by the c.d.f. of \([\nabla g(c)]^\top Y\), under Theorem 1.12(i).

- When \(\nabla g(c) = 0\), Theorem 1.12(i) indicates that the limiting distribution of \(a_n[g(X_n) - g(c)]\) is degenerated. In such cases the result in Theorem 1.12(ii) may be useful.
Proof (continued)

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\[
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Corollary 1.1 (the $\delta$-method)

Assume the conditions of Theorem 1.12. If $Y$ has the $N_k(0, \Sigma)$ distribution, then

$$a_n[g(X_n) - g(c)] \xrightarrow{d} N\left(0, [\nabla g(c)]^\top \Sigma \nabla g(c)\right).$$

Example 1.31

Let $\{X_n\}$ be a sequence of random variables satisfying

$$\sqrt{n}(X_n - c) \xrightarrow{d} N(0, 1).$$

Consider the function $g(x) = x^2$. If $c \neq 0$, then an application of Corollary 1.1 gives that

$$\sqrt{n}(X_n^2 - c^2) \xrightarrow{d} N(0, 4c^2).$$

If $c = 0$, $g'(c) = 0$ but $g''(c) = 2$. Hence, an application of Theorem 1.12(ii) gives that

$$nX_n^2 \xrightarrow{d} [N(0, 1)]^2,$$ 

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Example: Ratio estimator

$(X_1, Y_1),..., (X_n, Y_n)$ are iid bivariate random vectors with finite 2nd order moments

$\bar{X}_n = n^{-1} \sum_i X_i, \quad \bar{Y}_n = n^{-1} \sum_i Y_i$

$\mu_x = E(X_1), \mu_y = E(Y_1) \neq 0, \sigma_x^2 = \text{Var}(X_1), \sigma_y^2 = \text{Var}(Y_1), \sigma_{xy} = \text{Cov}(X_1, Y_1)$

By the CLT,

$$\sqrt{n}\left(\begin{pmatrix} \bar{X}_n \\ \bar{Y}_n \end{pmatrix} - \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}\right) \xrightarrow{d} N_2\left(0, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}\right)$$

By the $\delta$-method, $g(x, y) = x/y, \partial g/\partial x = y^{-1}, \partial g/\partial y = -xy^{-2}$

$$\sqrt{n}\left(\frac{\bar{X}_n}{\bar{Y}_n} - \frac{\mu_x}{\mu_y}\right) \xrightarrow{d} N(0, \sigma^2)$$

$$\sigma^2 = (\mu_y^{-1}, -\mu_x \mu_y^{-2}) \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \begin{pmatrix} \mu_y^{-1} \\ -\mu_x \mu_y^{-2} \end{pmatrix} = \frac{\sigma_x^2}{\mu_y^2} - \frac{\mu_x \sigma_{xy}}{\mu_y^3} + \frac{\mu_x^2 \sigma_y^2}{\mu_y^4}$$