The WLLN and SLLN may not be useful in approximating the distributions of (normalized) sums of independent random variables. We need to use the central limit theorem (CLT), which plays a fundamental role in statistical asymptotic theory.

**Theorem 1.15 (Lindeberg’s CLT)**

Let \( \{X_{nj}, j = 1, \ldots, k_n\} \) be independent random variables with \( k_n \to \infty \) as \( n \to \infty \) and
\[
0 < \sigma_n^2 = \text{var} \left( \sum_{j=1}^{k_n} X_{nj} \right) < \infty, \quad n = 1, 2, \ldots,
\]

If
\[
\frac{1}{\sigma_n^2} \sum_{j=1}^{k_n} E \left[ (X_{nj} - EX_{nj})^2 I_{\{|X_{nj} - EX_{nj}| > \epsilon \sigma_n\}} \right] \to 0 \quad \text{for any } \epsilon > 0, \quad (1)
\]

then
\[
\frac{1}{\sigma_n} \sum_{j=1}^{k_n} (X_{nj} - EX_{nj}) \to_d N(0, 1).
\]
The WLLN and SLLN may not be useful in approximating the distributions of (normalized) sums of independent random variables. We need to use the central limit theorem (CLT), which plays a fundamental role in statistical asymptotic theory.

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If

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\frac{1}{\sigma_n^2} \sum_{j=1}^{k_n} E \left[ (X_{nj} - EX_{nj})^2 I_{\{X_{nj} - EX_{nj} > \varepsilon \sigma_n\}} \right] \to 0 \quad \text{for any } \varepsilon > 0, \tag{1}
\]

then

\[
\frac{1}{\sigma_n} \sum_{j=1}^{k_n} (X_{nj} - EX_{nj}) \to_d N(0, 1).
\]
Proof

Considering \((X_{nj} - EX_{nj})/\sigma_n\), without loss of generality we may assume \(EX_{nj} = 0\) and \(\sigma_n^2 = 1\) in this proof.

Let \(t \in \mathbb{R}\) be given.

From the inequality

\[
|e^{-\frac{1}{2}tx} - (1 + \sqrt{-1}tx - t^2x^2/2)| \leq \min\{|tx|^2, |tx|^3\},
\]

the ch.f. of \(X_{nj}\) satisfies

\[
\left| \phi_{X_{nj}}(t) - \left(1 - t^2\sigma_{nj}^2/2\right) \right| \leq \mathbb{E} \left( \min\{|tX_{nj}|^2, |tX_{nj}|^3\} \right),
\]

where \(\sigma_{nj}^2 = \text{var}(X_{nj})\).

For any \(\varepsilon > 0\), the right-hand side of the previous expression is bounded by

\[
\mathbb{E}(|tX_{nj}|^3 I_{\{|X_{nj}|<\varepsilon\}}) + \mathbb{E}(|tX_{nj}|^2 I_{\{|X_{nj}|\geq\varepsilon\}}),
\]

which is bounded by

\[
\varepsilon |t|^3 \sigma_{nj}^2 + t^2 \mathbb{E}(X_{nj}^2 I_{\{|X_{nj}|\geq\varepsilon\}}).
\]
Proof (continued)

Summing over $j$ and using $\sigma_n^2 = 1$, we obtain that

$$\sum_{j=1}^{k_n} \left| \phi_{X_{nj}}(t) - \left(1 - t^2 \sigma_{nj}^2 / 2\right) \right| \leq \sum_{j=1}^{k_n} \{ \varepsilon |t|^3 \sigma_{nj}^2 + t^2 E(X_{nj}^2 I_{\{|X_{nj}|\geq \varepsilon\}}) \}$$

$$= \varepsilon |t|^3 + t^2 \sum_{j=1}^{k_n} E(X_{nj}^2 I_{\{|X_{nj}|\geq \varepsilon\}}) \to \varepsilon |t|^3$$

by condition (1).

Also by condition (1) and $\sigma_n^2 = 1$,

$$\max_{j \leq k_n} \frac{\sigma_{nj}^2}{\sigma_n^2} \leq \varepsilon^2 + \max_{j \leq k_n} E(X_{nj}^2 I_{\{|X_{nj}| > \varepsilon\}}) \to \varepsilon^2$$

Since $\varepsilon > 0$ is arbitrary and $t$ is fixed,

$$\sum_{j=1}^{k_n} \left| \phi_{X_{nj}}(t) - \left(1 - t^2 \sigma_{nj}^2 / 2\right) \right| \to 0$$

and

$$\lim_{n \to \infty} \max_{j \leq k_n} \frac{\sigma_{nj}^2}{\sigma_n^2} = 0. \quad (2)$$
Proof (continued)

This implies that $1 - t^2 \sigma_{nj}^2$ are all between 0 and 1 for large enough $n$. Using the inequality

$$\left| a_1 \cdots a_m - b_1 \cdots b_m \right| \leq \sum_{j=1}^{m} |a_j - b_j|$$

for any complex numbers $a_j$’s and $b_j$’s with $|a_j| \leq 1$ and $|b_j| \leq 1$, $j = 1, \ldots, m$, we obtain that

$$\left| \prod_{j=1}^{k_n} e^{-t^2 \sigma_{nj}^2 / 2} - \prod_{j=1}^{k_n} \left( 1 - t^2 \sigma_{nj}^2 / 2 \right) \right| \leq \sum_{j=1}^{k_n} \left| e^{-t^2 \sigma_{nj}^2 / 2} - \left( 1 - t^2 \sigma_{nj}^2 / 2 \right) \right|,$$

which is bounded by

$$t^4 \sum_{j=1}^{k_n} \sigma_{nj}^4 \leq t^4 \max_{j \leq k_n} \sigma_{nj}^2 \rightarrow 0,$$

since $|e^x - 1 - x| \leq x^2 / 2$ if $|x| \leq \frac{1}{2}$ and $\sum_{j=1}^{k_n} \sigma_{nj}^2 = \sigma_n^2 = 1.$
Proof (continued)

Then

\[
\left| \prod_{j=1}^{k_n} \phi_{X_{nj}}(t) - \prod_{j=1}^{k_n} e^{-t^2 \sigma_{nj}^2/2} \right| \leq \sum_{j=1}^{k_n} \left| \phi_{X_{nj}}(t) - e^{-t^2 \sigma_{nj}^2/2} \right| \\
\leq \sum_{j=1}^{k_n} \left| \phi_{X_{nj}}(t) - \left(1 - t^2 \sigma_{nj}^2/2\right) \right| \\
+ \sum_{j=1}^{k_n} \left| e^{-t^2 \sigma_{nj}^2/2} - \left(1 - t^2 \sigma_{nj}^2/2\right) \right| \\
\to 0
\]

as previously shown.
Thus,

\[
\prod_{j=1}^{k_n} \phi_{X_{nj}}(t) = \prod_{j=1}^{k_n} e^{-t^2 \sigma_{nj}^2/2} + o(1) = e^{-t^2/2} + o(1)
\]
i.e., the ch.f. of \( \sum_{j=1}^{k_n} X_{nj} \) converges to the ch.f. of \( N(0, 1) \) for every \( t \).

By Theorem 1.9(ii), the result follows.
Remarks

- Condition (1) is called Lindeberg’s condition.
- From the proof, Lindeberg’s condition implies (2), which is called Feller’s condition.
- Feller’s condition (2) means that all terms in the sum \( \sigma_n^2 = \sum_{j=1}^{k_n} \sigma_{nj}^2 \) are uniformly negligible as \( n \to \infty \).
- If Feller’s condition is assumed, then Lindeberg’s condition is not only sufficient but also necessary for the result in Theorem 1.15, which is the well-known Lindeberg-Feller CLT.
- A proof can be found in Billingsley (1995, pp. 359-361).
- Note that neither Lindeberg’s condition nor Feller’s condition is necessary for the result in Theorem 1.15 (Exercise 158).

Liapounov’s condition

A sufficient condition for Lindeberg’s condition is the following Liapounov’s condition, which is somewhat easier to verify:

\[
\frac{1}{\sigma_n^{2+\delta}} \sum_{j=1}^{k_n} E|X_{nj} - EX_{nj}|^{2+\delta} \to 0 \quad \text{for some } \delta > 0. \tag{3}
\]
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- From the proof, Lindeberg’s condition implies (2), which is called Feller’s condition.
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Example 1.33

Let $X_1, X_2, \ldots$ be independent random variables. Suppose that $X_i$ has the binomial distribution $Bi(p_i, 1)$, $i = 1, 2, \ldots$, and that $\sigma_n^2 = \sum_{i=1}^{n} \text{var}(X_i) = \sum_{i=1}^{n} p_i(1 - p_i) \to \infty$ as $n \to \infty$.

For each $i$, $EX_i = p_i$ and

$$E|X_i - EX_i|^3 = (1 - p_i)^3 p_i + p_i^3 (1 - p_i) \leq 2p_i(1 - p_i).$$

Hence $\sum_{i=1}^{n} E|X_i - EX_i|^3 \leq 2\sigma_n^2$, i.e., Liapounov’s condition (3) holds with $\delta = 1$.

Thus, by Theorem 1.15,

$$\frac{1}{\sigma_n} \sum_{i=1}^{n} (X_i - p_i) \to_d N(0, 1). \quad (4)$$

It can be shown (exercise) that the condition $\sigma_n \to \infty$ is also necessary for result (4).

The following are useful corollaries of Theorem 1.15 and Theorem 1.9(iii).
Example 1.33

Let $X_1, X_2, \ldots$ be independent random variables.
Suppose that $X_i$ has the binomial distribution $Bi(p_i, 1)$, $i = 1, 2, \ldots$, and that
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\sigma_n^2 = \sum_{i=1}^{n} \text{var}(X_i) = \sum_{i=1}^{n} p_i(1 - p_i) \to \infty \text{ as } n \to \infty.
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Hence $\sum_{i=1}^{n} E|X_i - EX_i|^3 \leq 2\sigma_n^2$, i.e., Liapounov’s condition (3) holds with $\delta = 1$.

Thus, by Theorem 1.15,

$$
\frac{1}{\sigma_n} \sum_{i=1}^{n} (X_i - p_i) \to_d N(0, 1).
$$

(4)

It can be shown (exercise) that the condition $\sigma_n \to \infty$ is also necessary for result (4).

The following are useful corollaries of Theorem 1.15 and Theorem 1.9(iii).
Corollary 1.2 (Multivariate CLT)
For i.i.d. random $k$-vectors $X_1, \ldots, X_n$ with a finite $\Sigma = \text{var}(X_1)$,
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - EX_1) \rightarrow_d N_k(0, \Sigma).
\]

Corollary 1.3
Let $X_{ni} \in \mathbb{R}^{m_i}$, $i = 1, \ldots, k_n$, be independent random vectors with $m_i \leq m$ (a fixed integer), $n = 1, 2, \ldots, k_n \rightarrow \infty$ as $n \rightarrow \infty$, and
\[
\inf_{i,n} \lambda_{-}[\text{var}(X_{ni})] > 0,
\]
where $\lambda_{-}[A]$ is the smallest eigenvalue of $A$.
Let $c_{ni} \in \mathbb{R}^{m_i}$ be vectors such that
\[
\lim_{n \rightarrow \infty} \left( \max_{1 \leq i \leq k_n} \|c_{ni}\|^2 / \sum_{i=1}^{k_n} \|c_{ni}\|^2 \right) = 0.
\]

(i) If $\sup_{i,n} E\|X_{ni}\|^{2+\delta} < \infty$ for some $\delta > 0$, then
\[
\sum_{i=1}^{k_n} c_{ni}^\tau (X_{ni} - EX_{ni}) / \left[ \sum_{i=1}^{k_n} \text{var}(c_{ni}^\tau X_{ni}) \right]^{1/2} \rightarrow_d N(0,1).
\]

(ii) If whenever $m_i = m_j$, $1 \leq i < j \leq k_n$, $n = 1, 2, \ldots$, $X_{ni}$ and $X_{nj}$ have the same distribution with $E\|X_{ni}\|^2 < \infty$, then (5) holds.
Corollary 1.2 (Multivariate CLT)

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Let \( X_{ni} \in \mathbb{R}^{m_i} \), \( i = 1, \ldots, k_n \), be independent random vectors with \( m_i \leq m \) (a fixed integer), \( n = 1, 2, \ldots, k_n \to \infty \) as \( n \to \infty \), and

\[
\inf_{i,n} \lambda_\cdot [\text{var}(X_{ni})] > 0,
\]

where \( \lambda_\cdot [A] \) is the smallest eigenvalue of \( A \). Let \( c_{ni} \in \mathbb{R}^{m_i} \) be vectors such that

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\lim_{n \to \infty} \left( \max_{1 \leq i \leq k_n} \|c_{ni}\|^2 / \sum_{i=1}^{k_n} \|c_{ni}\|^2 \right) = 0.
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(i) If \( \sup_{i,n} E \|X_{ni}\|^{2+\delta} < \infty \) for some \( \delta > 0 \), then

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\sum_{i=1}^{k_n} c_{ni}^\tau (X_{ni} - EX_{ni}) / \left[ \sum_{i=1}^{k_n} \text{var}(c_{ni}^\tau X_{ni}) \right]^{1/2} \to_d N(0,1). \tag{5}
\]

(ii) If whenever \( m_i = m_j, 1 \leq i < j \leq k_n, n = 1, 2, \ldots \), \( X_{ni} \) and \( X_{nj} \) have the same distribution with \( E \|X_{ni}\|^2 < \infty \), then (5) holds.
Remarks

- Proving Corollary 1.3 is a good exercise.
- Applications of these corollaries can be found in later chapters.
- More results on the CLT can be found, for example, in Serfling (1980) and Shorack and Wellner (1986).

More on Pólya’s theorem

Let $Y_n$ be a sequence of random variables, \{\mu_n\} and \{\sigma_n\} be sequences of real numbers such that $\sigma_n > 0$ for all $n$, and

$$(Y_n - \mu_n)/\sigma_n \to_d N(0,1).$$

Then, by Proposition 1.16,

$$\lim_{n \to \infty} \sup_x \left| F_{(Y_n - \mu_n)/\sigma_n}(x) - \Phi(x) \right| = 0,$$

where $\Phi$ is the c.d.f. of $N(0,1)$. 
Remarks

- Proving Corollary 1.3 is a good exercise.
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where $\Phi$ is the c.d.f. of $N(0, 1)$. 

(6)
Asymptotic normality

(6) implies that for any sequence of real numbers \( \{c_n\} \),

\[
\lim_{n \to \infty} |P(Y_n \leq c_n) - \Phi\left(\frac{c_n - \mu_n}{\sigma_n}\right)| = 0,
\]
i.e., \( P(Y_n \leq c_n) \) can be approximated by \( \Phi\left(\frac{c_n - \mu_n}{\sigma_n}\right) \), regardless of whether \( \{c_n\} \) has a limit.

Since \( \Phi\left(\frac{t - \mu_n}{\sigma_n}\right) \) is the c.d.f. of \( N(\mu_n, \sigma_n^2) \), \( Y_n \) is said to be asymptotically distributed as \( N(\mu_n, \sigma_n^2) \) or simply asymptotically normal.

Examples

- For example, \( \sum_{i=1}^{kn} c_{ni}^\tau X_{ni} \) in Corollary 1.3 is asymptotically normal.
- This can be extended to random vectors.
  For example, \( \sum_{i=1}^{n} X_i \) in Corollary 1.2 is asymptotically distributed as \( N_k(nEX_1, n\Sigma) \).
Asymptotic normality

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Examples

- For example, \( \sum_{i=1}^{k_n} c_{ni}^{T} X_{ni} \) in Corollary 1.3 is asymptotically normal.

- This can be extended to random vectors.
  For example, \( \sum_{i=1}^{n} X_i \) in Corollary 1.2 is asymptotically distributed as \( N_k(nEX_1, n\Sigma) \).