### Application

- One or a series of random experiments is performed.
- Some data from the experiment(s) are collected.
- Planning experiments and collecting data (not discussed in the textbook).
- Data analysis: extract information from the data, interpret the results, and draw some conclusions.

### Descriptive data analysis

- Summary measures of the data, such as the mean, median, range, standard deviation, etc., and some graphical displays, such as the histogram and box-and-whisker diagram, etc.
- It is simple and requires almost no assumptions, but may not allow us to gain enough insight into the problem.
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We focus on more sophisticated methods of analyzing data: *statistical inference* and *decision theory*.

The data set is a realization of a random element defined on a probability space \((\Omega, \mathcal{F}, P)\).

*\(P\) is called the *population*. 

The data set or the random element that produces the data is called a *sample* from *\(P\)*.

The size of the data set is called the *sample size*.

---

**Our task**

- A population *\(P\)* is *known* iff *\(P(A)\)* is a known value for every event *\(A \in \mathcal{F}\).*
- In a statistical problem, the population *\(P\)* is at least partially unknown.
- We would like to deduce some properties of *\(P\)* based on the available sample.
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Read Examples 2.1-2.3

### Statistical model

- A **statistical model** is a set of assumptions on the population $P$ and is often postulated to make the analysis possible or easy.
- Postulated models are often based on knowledge of the problem under consideration.

### Definition 2.1

A set of probability measures $P_{\theta}$ on $(\Omega, \mathcal{F})$ indexed by a **parameter** $\theta \in \Theta$ is said to be a **parametric family** iff $\Theta \subset \mathbb{R}^d$ for some fixed positive integer $d$ and each $P_{\theta}$ is a known probability measure when $\theta$ is known.

The set $\Theta$ is called the **parameter space** and $d$ is called its **dimension**.

### Parametric model

The population $P$ is in a parametric family $\mathcal{P} = \{P_{\theta} : \theta \in \Theta\}$.
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- \( \mathcal{P} = \{ P_\theta : \theta \in \Theta \} \) is *identifiable* iff \( \theta_1 \neq \theta_2 \) and \( \theta_i \in \Theta \) imply \( P_{\theta_1} \neq P_{\theta_2} \).

- In most cases an identifiable parametric family can be obtained through reparameterization.

- A family of populations \( \mathcal{P} \) is dominated by \( \nu \) (a \( \sigma \)-finite measure) if \( P \ll \nu \) for all \( P \in \mathcal{P} \).

- \( \mathcal{P} \) can be identified by the family of densities \( \left\{ \frac{dP}{d\nu} : P \in \mathcal{P} \right\} \) or \( \left\{ \frac{dP_\theta}{d\nu} : \theta \in \Theta \right\} \).

**Parametric methods**

Methods designed for parametric models

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Semi-parametric models and methods

Example (The \( k \)-dimensional normal family)

\[
\mathcal{P} = \{ N_k(\mu, \Sigma) : \mu \in \mathbb{R}^k, \Sigma \in \mathcal{M}_k \},
\]

where \( \mathcal{M}_k \) is a collection of \( k \times k \) symmetric positive definite matrices. This family is a parametric family dominated by the Lebesgue measure on \( \mathbb{R}^k \).

When \( k = 1 \), \( \mathcal{P} = \{ N(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0 \} \).

Examples of nonparametric family on \((\mathbb{R}^k, \mathcal{B}^k)\)

- The joint c.d.f.’s are continuous.
- The joint c.d.f.’s have finite moments of order \( \leq \) a fixed integer.
- The joint c.d.f.’s have p.d.f.’s (e.g., Lebesgue p.d.f.’s).
- \( k = 1 \) and the c.d.f.’s are symmetric.
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Nonparametric methods
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Statistics and their distributions

- Our data set is a realization of a sample (random vector) $X$ from an unknown population $P$
- Statistic $T(X)$: A measurable function $T$ of $X$; $T(X)$ is a known value whenever $X$ is known.
- Statistical analyses are based on various statistics, for various purposes.
- $X$ itself is a statistic, but it is a trivial statistic.
- The range of a nontrivial statistic $T(X)$ is usually simpler than that of $X$.
- For example, $X$ may be a random $n$-vector and $T(X)$ may be a random $p$-vector with a $p$ much smaller than $n$.
- $\sigma(T(X)) \subset \sigma(X)$ and the two $\sigma$-fields are the same iff $T$ is one-to-one.
- Usually $\sigma(T(X))$ simplifies $\sigma(X)$, i.e., a statistic provides a “reduction” of the $\sigma$-field.
The “information” within a statistic

- The “information” within the statistic $T(X)$ concerning the unknown distribution of $X$ is contained in the $\sigma$-field $\sigma(T(X))$.
- $S$ is any other statistic for which $\sigma(S(X)) = \sigma(T(X))$. By Lemma 1.2, $S$ is a measurable function of $T$, and $T$ is a measurable function of $S$. Thus, once the value of $S$ (or $T$) is known, so is the value of $T$ (or $S$).
- It is not the particular values of a statistic that contain the information, but the generated $\sigma$-field of the statistic.
- Values of a statistic may be important for other reasons.

Distribution of a statistic

- A statistic $T(X)$ is a random element.
- If the distribution of $X$ is unknown, then the distribution of $T$ may also be unknown, although $T$ is a known function.
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**Distribution of a statistic**

- Finding the form of the distribution of $T$ is one of the major problems in statistical inference and decision theory.
- Since $T$ is a transformation of $X$, tools we learn in Chapter 1 for transformations may be useful in finding the distribution or an approximation to the distribution of $T(X)$.

**Example 2.8.**

Let $X_1, \ldots, X_n$ be i.i.d. random variables having a common distribution $P$ and $X = (X_1, \ldots, X_n)$. The sample mean

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

and the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

are two commonly used statistics.
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**Example 2.28 (continued)**

Can we find the joint or the marginal distributions of $\bar{X}$ and $S^2$? It depends on how much we know about $P$.

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<th>Moments of $\bar{X}$ and $S^2$</th>
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- Even if the form of $\mu$ is known, $\mu(\theta)$ is till unknown when $\theta$ is unknown.
- If $P$ has a finite variance $\sigma^2$, then $\text{var}(\bar{X}) = \sigma^2/n$, which equals $\sigma^2(\theta)/n$ for some function $\sigma^2(.)$ if $P$ is in a parametric family.
- With a finite $\sigma^2 = \text{var}(X_1)$, we can also obtain that $ES^2 = \sigma^2$.
- With a finite $E|X_1|^3$, we can obtain $E(\bar{X})^3$ and $\text{Cov}(\bar{X}, S^2)$.
- With a finite $E(X_1)^4$, we can obtain $\text{var}(S^2)$ (exercise).
Example 2.28 (continued)

The distribution of $\bar{X}$

If $P$ is in a parametric family, we can often find the distribution of $\bar{X}$. For example:

- $\bar{X}$ is $N(\mu, \sigma^2/n)$ if $P$ is $N(\mu, \sigma^2)$;
- $n\bar{X}$ has the gamma distribution $\Gamma(n, \theta)$ if $P$ is the exponential distribution $E(0, \theta)$;
- See Example 1.20 and some exercises in §1.6.

One can use the CLT to obtain an approximation to the distribution of $\bar{X}$.

Applying Corollary 1.2 (for the case of $k = 1$), we obtain that $\sqrt{n}(\bar{X} - \mu) \rightarrow_d N(0, \sigma^2)$, where $\mu$ and $\sigma^2$ are the mean and variance of $P$, respectively, and are assumed to be finite.

The distribution of $\bar{X}$ can be approximated by $N(\mu, \sigma^2/n)$.
Example 2.28 (continued)

The distribution of $S^2$

If $P$ is $N(\mu, \sigma^2)$, then $(n - 1)S^2/\sigma^2$ has the chi-square distribution $\chi^2_{n-1}$ (see Example 2.18).

An approximate distribution for $S^2$ can be obtained from the approximate joint distribution of $\bar{X}$ and $S^2$ discussed next.

Joint distribution of $\bar{X}$ and $S^2$

If $P$ is $N(\mu, \sigma^2)$, then $\bar{X}$ and $S^2$ are independent (Example 2.18). Hence, the joint distribution of $(\bar{X}, S^2)$ is the product of the marginal distributions of $\bar{X}$ and $S^2$ given in the previous discussion.

Without the normality assumption, an approximate joint distribution can be obtained.
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Without the normality assumption, an approximate joint distribution can be obtained.
Example 2.28 (continued)

Assume that \( \mu = EX_1, \sigma^2 = \text{var}(X_1), \) and \( E|X_1|^4 \) are finite. Let \( Y_i = (X_i - \mu, (X_i - \mu)^2), \) \( i = 1, \ldots, n. \)
\( Y_1, \ldots, Y_n \) are i.i.d. random 2-vectors with \( EY_1 = (0, \sigma^2) \) and variance-covariance matrix
\[
\Sigma = \begin{pmatrix}
\sigma^2 & E(X_1 - \mu)^3 \\
E(X_1 - \mu)^3 & E(X_1 - \mu)^4 - \sigma^4
\end{pmatrix}.
\]
Note that \( \bar{Y} = n^{-1} \sum_{i=1}^{n} Y_i = (\bar{X} - \mu, \tilde{S}^2), \) where \( \tilde{S}^2 = n^{-1} \sum_{i=1}^{n} (X_i - \mu)^2. \)

Applying the CLT (Corollary 1.2) to \( Y_i \)'s, we obtain that
\[ \sqrt{n}(\bar{X} - \mu, \tilde{S}^2 - \sigma^2) \to_d N_2(0, \Sigma). \]

Since
\[ S^2 = \frac{n}{n-1} \left[ \tilde{S}^2 - (\bar{X} - \mu)^2 \right] \]
and \( \bar{X} \to_{a.s.} \mu \) (the SLLN), an application of Slutsky’s theorem leads to
\[ \sqrt{n}(\bar{X} - \mu, S^2 - \sigma^2) \to_d N_2(0, \Sigma). \]
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Assume that $\mu = EX_1$, $\sigma^2 = \text{var}(X_1)$, and $E|X_1|^4$ are finite.
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Note that $\bar{Y} = n^{-1} \sum_{i=1}^n Y_i = (\bar{X} - \mu, \tilde{S}^2)$, where $\tilde{S}^2 = n^{-1} \sum_{i=1}^n (X_i - \mu)^2$.

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\sqrt{n}(\bar{X} - \mu, \tilde{S}^2 - \sigma^2) \xrightarrow{d} \mathcal{N}_2(0, \Sigma).
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Example 2.9 (Order statistics)

Let $X = (X_1, ..., X_n)$ with i.i.d. random components. Let $X_{(i)}$ be the $i$th smallest value of $X_1, ..., X_n$. The statistics $X_{(1)}, ..., X_{(n)}$ are called the order statistics. Order statistics is a set of very useful statistics in addition to the sample mean and variance.

Suppose that $X_i$ has a c.d.f. $F$ having a Lebesgue p.d.f. $f$. Then the joint Lebesgue p.d.f. of $X_{(1)}, ..., X_{(n)}$ is

$$g(x_1, x_2, ..., x_n) = \begin{cases} 
n! f(x_1)f(x_2)\cdots f(x_n) & x_1 < x_2 < \cdots < x_n \\
0 & \text{otherwise.} \end{cases}$$

The joint Lebesgue p.d.f. of $X_{(i)}$ and $X_{(j)}$, $1 \leq i < j \leq n$, is

$$g_{i,j}(x, y) = \begin{cases} 
n![F(x)]^{i-1}[F(y) - F(x)]^{j-i-1}[1-F(y)]^{n-j}f(x)f(y) & x < y \\
0 & \text{otherwise} \end{cases}$$

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The joint Lebesgue p.d.f. of \( X(i) \) and \( X(j), 1 \leq i < j \leq n \), is

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g_{i,j}(x, y) = \begin{cases} 
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   0 & \text{otherwise}
\end{cases}
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and the Lebesgue p.d.f. of \( X(i) \) is

\[
g_i(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x).
\]