Theorem 3.3 (Cramér-Rao lower bound)

Let \( X = (X_1, \ldots, X_n) \) be a sample from \( P \in \mathcal{P} = \{P_\theta : \theta \in \Theta\} \), where \( \Theta \) is an open set in \( \mathbb{R}^k \).

Suppose that \( T(X) \) is an estimator with \( E[T(X)] = g(\theta) \) being a differentiable function of \( \theta \); \( P_\theta \) has a p.d.f. \( f_\theta \) w.r.t. a measure \( \nu \) for all \( \theta \in \Theta \); and \( f_\theta \) is differentiable as a function of \( \theta \) and satisfies

\[
\frac{\partial}{\partial \theta} \int h(x)f_\theta(x)\,d\nu = \int h(x)\frac{\partial}{\partial \theta}f_\theta(x)\,d\nu, \quad \theta \in \Theta, \tag{1}
\]

for \( h(x) \equiv 1 \) and \( h(x) = T(x) \).

Then

\[
\text{Var}(T(X)) \geq \left[ \frac{\partial}{\partial \theta} g(\theta) \right]^\tau \left[ I(\theta) \right]^{-1} \frac{\partial}{\partial \theta} g(\theta), \tag{2}
\]

where

\[
I(\theta) = E \left\{ \frac{\partial}{\partial \theta} \log f_\theta(X) \left[ \frac{\partial}{\partial \theta} \log f_\theta(X) \right]^{\tau} \right\}
\]

is assumed to be positive definite for any \( \theta \in \Theta \).
Discussion

Suppose that we have a lower bound for the variances of all unbiased estimators of $\vartheta$.
If there is an unbiased estimator $T$ of $\vartheta$ whose variance is always the same as the lower bound, then $T$ is a UMVUE of $\vartheta$.
Although this is not an effective way to find UMVUE’s, it provides a way of assessing the performance of UMVUE’s.

Proof of Theorem 3.3

We prove the univariate case ($k = 1$) only.
When $k = 1$, (2) reduces to

$$\text{Var}(T(X)) \geq \frac{[g'(\theta)]^2}{E \left[ \frac{\partial}{\partial \theta} \log f_{\theta}(X) \right]^2}.$$  

From the Cauchy-Schwartz inequality, we only need to show that

$$E \left[ \frac{\partial}{\partial \theta} \log f_{\theta}(X) \right]^2 = \text{Var} \left( \frac{\partial}{\partial \theta} \log f_{\theta}(X) \right)$$
Proof of Theorem 3.3 (continued)

and

\[ g'(\theta) = \text{Cov} \left( T(X), \frac{\partial}{\partial \theta} \log f_\theta(X) \right). \]

From condition (1) with \( h(x) = 1, \)

\[ E \left[ \frac{\partial}{\partial \theta} \log f_\theta(X) \right] = \int \frac{\partial}{\partial \theta} f_\theta(X) d\nu = \frac{\partial}{\partial \theta} \int f_\theta(X) d\nu = 0. \]

From condition (1) with \( h(x) = T(x), \)

\[ E \left[ T(X) \frac{\partial}{\partial \theta} \log f_\theta(X) \right] = \int T(x) \frac{\partial}{\partial \theta} f_\theta(X) d\nu = \frac{\partial}{\partial \theta} \int T(x) f_\theta(X) d\nu, \]

which = \( g'(\theta) \).

The \( k \times k \) matrix

\[ I(\theta) = E \left\{ \frac{\partial}{\partial \theta} \log f_\theta(X) \left[ \frac{\partial}{\partial \theta} \log f_\theta(X) \right]^\tau \right\} \]

is called the \textit{Fisher information matrix}. 
The greater $I(\theta)$ is, the easier it is to distinguish $\theta$ from neighboring values and, therefore, the more accurately $\theta$ can be estimated. Thus, $I(\theta)$ is a measure of the information that $X$ contains about $\theta$. The inequality in (2) is called *information inequalities*.

The following result is helpful in finding the Fisher information matrix.

**Proposition 3.1**

(i) If $X$ and $Y$ are independent with the Fisher information matrices $I_X(\theta)$ and $I_Y(\theta)$, respectively, then the Fisher information about $\theta$ contained in $(X, Y)$ is $I_X(\theta) + I_Y(\theta)$. In particular, if $X_1, \ldots, X_n$ are i.i.d. and $I_1(\theta)$ is the Fisher information about $\theta$ contained in a single $X_i$, then the Fisher information about $\theta$ contained in $X_1, \ldots, X_n$ is $nI_1(\theta)$.

(ii) Suppose that $X$ has the p.d.f. $f_\theta$ that is twice differentiable in $\theta$ and that (1) holds with $h(x) \equiv 1$ and $f_\theta$ replaced by $\partial f_\theta / \partial \theta$. Then

$$I(\theta) = -E \left[ \frac{\partial^2}{\partial \theta \partial \theta^\tau} \log f_\theta(X) \right].$$
Proof

Result (i) follows from the independence of $X$ and $Y$ and the definition of the Fisher information.

Result (ii) follows from the equality

$$\frac{\partial^2}{\partial \theta \partial \theta^\tau} \log f_\theta(X) = \frac{\partial^2}{\partial \theta \partial \theta^\tau} f_\theta(X) \frac{f_\theta(X)}{f_\theta(X)} - \frac{\partial}{\partial \theta} \log f_\theta(X) \left[ \frac{\partial}{\partial \theta} \log f_\theta(X) \right]^\tau.$$ 

Example 3.9

Let $X_1, ..., X_n$ be i.i.d. with the Lebesgue p.d.f. $\frac{1}{\sigma} f \left( \frac{x - \mu}{\sigma} \right)$, where $f(x) > 0$ and $f'(x)$ exists for all $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $\sigma > 0$ (a location-scale family).

Let $\theta = (\mu, \sigma)$. Then, the Fisher information about $\theta$ contained in $X_1, ..., X_n$ is (exercise)

$$I(\theta) = \frac{n}{\sigma^2} \left( \begin{array}{ccc} cc \int \frac{[f'(x)]^2}{f(x)} \, dx & \int \frac{f'(x)[xf'(x)+f(x)]}{f(x)} \, dx \\ \int \frac{f'(x)[xf'(x)+f(x)]}{f(x)} \, dx & \int \frac{[xf'(x)+f(x)]^2}{f(x)} \, dx \end{array} \right).$$
Remarks

- Note that $l(\theta)$ depends on the particular parameterization.
- If $\theta = \psi(\eta)$ and $\psi$ is differentiable, then the Fisher information that $X$ contains about $\eta$ is
  \[
  \frac{\partial}{\partial \eta} \psi(\eta) l(\psi(\eta)) \left[ \frac{\partial}{\partial \eta} \psi(\eta) \right]^\tau.
  \]
- However, the Cramér-Rao lower bound in (2) is not affected by any one-to-one reparameterization.
- If we use inequality (2) to find a UMVUE $T(X)$, then we obtain a formula for $\text{Var}(T(X))$ at the same time.
- On the other hand, the Cramér-Rao lower bound in (2) is typically not sharp.
- Under some regularity conditions, the Cramér-Rao lower bound is attained iff $f_\theta$ is in an exponential family; see Propositions 3.2 and 3.3 and the discussion in Lehmann (1983, p. 123).
- Some improved information inequalities are available (see, e.g., Lehmann (1983, Sections 2.6 and 2.7)).
Proposition 3.2.

Suppose that the distribution of $X$ is from an exponential family \( \{ f_\theta : \theta \in \Theta \} \), i.e., the p.d.f. of $X$ w.r.t. a $\sigma$-finite measure is

\[
f_\theta(x) = \exp\{ [\eta(\theta)]^T T(x) - \xi(\theta) \} c(x),
\]

where $\Theta$ is an open subset of $\mathbb{R}^k$.

(i) The regularity condition (1) is satisfied for any $h$ with $E|h(X)| < \infty$ and

\[
I(\theta) = -E \left[ \frac{\partial^2}{\partial \theta \partial \theta^T} \log f_\theta(X) \right].
\]

(ii) If $I(\eta)$ is the Fisher information matrix for the natural parameter $\eta$, then the variance-covariance matrix $\text{Var}(T) = I(\eta)$.

(iii) If $I(\vartheta)$ is the Fisher information matrix for the parameter $\vartheta = E[T(X)]$, then $\text{Var}(T) = [I(\vartheta)]^{-1}$. 
Proof

(i) This is a direct consequence of Theorem 2.1.

(ii) The p.d.f. under the natural parameter $\eta$ is

$$f_{\eta}(x) = \exp \left\{ \eta^\tau T(x) - \zeta(\eta) \right\} c(x).$$

From Theorem 2.1, $E[T(X)] = \frac{\partial}{\partial \eta} \zeta(\eta)$.

The result follows from

$$\frac{\partial}{\partial \eta} \log f_{\eta}(x) = T(x) - \frac{\partial}{\partial \eta} \zeta(\eta).$$

(iii) Since $\vartheta = E[T(X)] = \frac{\partial}{\partial \eta} \zeta(\eta)$,

$$I(\eta) = \frac{\partial \vartheta}{\partial \eta} I(\vartheta) \left( \frac{\partial \vartheta}{\partial \eta} \right)^\tau = \frac{\partial^2}{\partial \eta \partial \eta^\tau} \zeta(\eta) I(\vartheta) \left[ \frac{\partial^2}{\partial \eta \partial \eta^\tau} \zeta(\eta) \right]^\tau.$$

By Theorem 2.1 and the result in (ii),

$$\frac{\partial^2}{\partial \eta \partial \eta^\tau} \zeta(\eta) = \text{Var}(T) = I(\eta).$$

Hence

$$I(\vartheta) = [I(\eta)]^{-1} I(\eta) [I(\eta)]^{-1} = [I(\eta)]^{-1} = [\text{Var}(T)]^{-1}.$$

A direct consequence of Proposition 3.2(ii) is that the variance of any linear function of $T$ in (3) attains the Cramér-Rao lower bound.

The following result gives a necessary condition for $\text{Var}(U(X))$ of an estimator $U(X)$ to attain the Cramér-Rao lower bound.

**Proposition 3.3**

Assume that the conditions in Theorem 3.3 hold with $T(X)$ replaced by $U(X)$ and that $\Theta \subset \mathbb{R}$.

(i) If $\text{Var}(U(X))$ attains the Cramér-Rao lower bound in (2), then

$$a(\theta)[U(X) - g(\theta)] = g'(\theta) \frac{\partial}{\partial \theta} \log f_{\theta}(X) \quad \text{a.s.} \ P_{\theta}$$

for some function $a(\theta)$, $\theta \in \Theta$.

(ii) Let $f_{\theta}$ and $T$ be given by (3).
If $\text{Var}(U(X))$ attains the Cramér-Rao lower bound, then $U(X)$ is a linear function of $T(X)$ a.s. $P_{\theta}$, $\theta \in \Theta$. 
Example 3.10

Let $X_1, \ldots, X_n$ be i.i.d. from the $N(\mu, \sigma^2)$ distribution with an unknown $\mu \in \mathbb{R}$ and a known $\sigma^2$.

Let $f_\mu$ be the joint distribution of $X = (X_1, \ldots, X_n)$.

Then

$$\frac{\partial}{\partial \mu} \log f_\mu(X) = \sum_{i=1}^{n} \frac{(X_i - \mu)}{\sigma^2}.$$

Thus, $I(\mu) = n/\sigma^2$.

Consider the estimation of $\mu$.

It is obvious that $\text{Var}(\bar{X})$ attains the Cramér-Rao lower bound in (2).

Consider now the estimation of $\var = \mu^2$.

Since $E\bar{X}^2 = \mu^2 + \sigma^2/n$, the UMVUE of $\var$ is $h(\bar{X}) = \bar{X}^2 - \sigma^2/n$.

A straightforward calculation shows that

$$\text{Var}(h(\bar{X})) = \frac{4\mu^2 \sigma^2}{n} + \frac{2\sigma^4}{n^2}.$$
Example 3.10 (continued)

On the other hand, the Cramér-Rao lower bound in this case is $4\mu^2\sigma^2/n$. Hence $\text{Var}(h(\bar{X}))$ does not attain the Cramér-Rao lower bound. The difference is $2\sigma^4/n^2$.

Remarks

- Condition (1) is a key regularity condition for the results in Theorem 3.3 and Proposition 3.3.
- If $f_\theta$ is not in an exponential family, then (1) has to be checked.
- Typically, it does not hold if the set $\{x : f_\theta(x) > 0\}$ depends on $\theta$ (Exercise 37).
- More discussions can be found in Pitman (1979).