Lecture 28: UMVUE: a necessary and sufficient condition

When a complete and sufficient statistic is not available, it is usually very difficult to derive a UMVUE. In some cases, the following result can be applied, if we have enough knowledge about unbiased estimators of 0.

Theorem 3.2

Let $\mathcal{U}$ be the set of all unbiased estimators of $\vartheta$ with finite variances and $T$ be an unbiased estimator of $\vartheta$ with $E(T^2) < \infty$.

(i) A necessary and sufficient condition for $T(X)$ to be a UMVUE of $\vartheta$ is that $E[T(X)U(X)] = 0$ for any $U \in \mathcal{U}$ and any $P \in \mathcal{P}$.

(ii) Suppose that $T = h(\tilde{T})$, where $\tilde{T}$ is a sufficient statistic for $P \in \mathcal{P}$ and $h$ is a Borel function. Let $\mathcal{U}_{\tilde{T}}$ be the subset of $\mathcal{U}$ consisting of Borel functions of $\tilde{T}$. Then a necessary and sufficient condition for $T$ to be a UMVUE of $\vartheta$ is that $E[T(X)U(X)] = 0$ for any $U \in \mathcal{U}_{\tilde{T}}$ and any $P \in \mathcal{P}$. 
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Let $\mathcal{U}_{\tilde{T}}$ be the subset of $\mathcal{U}$ consisting of Borel functions of $\tilde{T}$. Then a necessary and sufficient condition for $T$ to be a UMVUE of $\vartheta$ is that $E[T(X)U(X)] = 0$ for any $U \in \mathcal{U}_{\tilde{T}}$ and any $P \in \mathcal{P}$.
Proof of Theorem 3.2(i)

Suppose that $T$ is a UMVUE of $\vartheta$. Then $T_c = T + cU$, where $U \in \mathcal{U}$ and $c$ is a fixed constant, is also unbiased for $\vartheta$ and, thus,

$$\text{Var}(T_c) \geq \text{Var}(T) \quad c \in \mathbb{R}, \ P \in \mathcal{P},$$

which is the same as

$$c^2 \text{Var}(U) + 2c \text{Cov}(T, U) \geq 0 \quad c \in \mathbb{R}, \ P \in \mathcal{P}.$$ 

This is impossible unless $\text{Cov}(T, U) = E(TU) = 0$ for any $P \in \mathcal{P}$.

Suppose now $E(TU) = 0$ for any $U \in \mathcal{U}$ and $P \in \mathcal{P}$. Let $T_0$ be another unbiased estimator of $\vartheta$ with $\text{Var}(T_0) < \infty$. Then $T - T_0 \in \mathcal{U}$ and, hence,

$$E[T(T - T_0)] = 0 \quad P \in \mathcal{P},$$

which with the fact that $ET = ET_0$ implies that

$$\text{Var}(T) = \text{Cov}(T, T_0) \quad P \in \mathcal{P}.$$ 

Note that $[\text{Cov}(T, T_0)]^2 \leq \text{Var}(T) \text{Var}(T_0)$. Hence $\text{Var}(T) \leq \text{Var}(T_0)$ for any $P \in \mathcal{P}$. 
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Proof of Theorem 3.2(ii)

It suffices to show that $E(TU) = 0$ for any $U \in \mathcal{U}$ and $P \in \mathcal{P}$ implies that $E(TU) = 0$ for any $U \in \mathcal{U}$ and $P \in \mathcal{P}$

Let $U \in \mathcal{U}$.
Then $E(U|\tilde{T}) \in \mathcal{U}$, and the result follows from the fact that $T = h(\tilde{T})$ and

$$E(TU) = E[E(TU|\tilde{T})] = E[E(h(\tilde{T})U|\tilde{T})] = E[h(\tilde{T})E(U|\tilde{T})].$$

Theorem 3.2 can be used

- to find a UMVUE,
- to check whether a particular estimator is a UMVUE, and
- to show the nonexistence of any UMVUE.

If there is a sufficient statistic, then by Rao-Blackwell’s theorem, we only need to focus on functions of the sufficient statistic and, hence, Theorem 3.2(ii) is more convenient to use.
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As a consequence of Theorem 3.2, we have the following useful result.

**Corollary 3.1**

(i) Let $T_j$ be a UMVUE of $\vartheta_j$, $j = 1, \ldots, k$, where $k$ is a fixed positive integer.
Then $\sum_{j=1}^{k} c_j T_j$ is a UMVUE of $\vartheta = \sum_{j=1}^{k} c_j \vartheta_j$ for any constants $c_1, \ldots, c_k$.

(ii) Let $T_1$ and $T_2$ be two UMVUE’s of $\vartheta$.
Then $T_1 = T_2$ a.s. $P$ for any $P \in \mathcal{P}$.

**Proof**

(i) Obviously, $\sum_{j=1}^{k} c_j T_j$ is a unbiased for $\vartheta = \sum_{j=1}^{k} c_j \vartheta_j$
For each $j$,

$$E(T_j U) = 0, \quad U \in \mathcal{U}$$

Then

$$E \left[ \left( \sum_{j=1}^{k} c_j T_j \right) U \right] = \sum_{j=1}^{k} c_j E(T_j U) = 0, \quad U \in \mathcal{U}$$
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Proof (continued)

(ii) Let $T_1$ and $T_2$ be two UMVUE’s of $\vartheta$. Then $T_1 - T_2 \in \mathcal{U}$ and

$$E[T_j(T_1 - T_2)] = 0 \quad j = 1, 2.$$ 

Then

$$E(T_1 - T_2)^2 = E[T_1(T_1 - T_2)] - E[T_2(T_1 - T_2)] = 0$$

Hence, $T_1 = T_2$ a.s. $P$ for any $P \in \mathcal{P}$.

Example 3.7

Let $X_1, \ldots, X_n$ be i.i.d. from the uniform distribution on the interval $(0, \theta)$. In Example 3.1, $(1 + n^{-1})X_{(n)}$ is shown to be the UMVUE for $\theta$ when the parameter space is $\Theta = (0, \infty)$. Suppose now that $\Theta = [1, \infty)$. Then $X_{(n)}$ is not complete, although it is still sufficient for $\theta$. Thus, Theorem 3.1 does not apply to $X_{(n)}$. 

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Example 3.7

Let $X_1, \ldots, X_n$ be i.i.d. from the uniform distribution on the interval $(0, \theta)$.
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the parameter space is $\Theta = (0, \infty)$.
Suppose now that $\Theta = [1, \infty)$.

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Thus, Theorem 3.1 does not apply to $X_{(n)}$. 
Example 3.7 (continued)

We now illustrate how to use Theorem 3.2(ii) to find a UMVUE of $\theta$.

Let $U(X(n))$ be an unbiased estimator of 0.

Since $X(n)$ has the Lebesgue p.d.f. $n\theta^{-n}x^{n-1}I_{(0,\theta)}(x)$,

$$0 = \int_0^1 U(x)x^{n-1}dx + \int_1^\theta U(x)x^{n-1}dx \quad \text{for all } \theta \geq 1.$$

This implies that $U(x) = 0$ a.e. Lebesgue measure on $[1, \infty)$ and

$$\int_0^1 U(x)x^{n-1}dx = 0.$$

Consider $T = h(X(n))$.

To have $E(TU) = 0$, we must have

$$\int_0^1 h(x)U(x)x^{n-1}dx = 0.$$

Thus, we may consider the following function:

$$h(x) = \begin{cases} 
c & 0 \leq x \leq 1 \\
bx & x > 1, \end{cases}$$

where $c$ and $b$ are some constants.
Example 3.7 (continued)

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This implies that $U(x) = 0$ a.e. Lebesgue measure on $[1, \infty)$ and $\int_0^1 U(x)x^{n-1} \, dx = 0$.

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We now illustrate how to use Theorem 3.2(ii) to find a UMVUE of \( \theta \).

Let \( U(X_{(n)}) \) be an unbiased estimator of 0.

Since \( X_{(n)} \) has the Lebesgue p.d.f. \( n\theta^{-n}x^{n-1}I_{(0,\theta)}(x), \)

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Example 3.7 (continued)

From the previous discussion,

$$E[h(X_n)U(X_n)] = 0, \quad \theta \geq 1.$$ 

Since $E[h(X_n)] = \theta$, we obtain that

$$\theta = cP(X_n \leq 1) + bE[X_n I_{(1,\infty)}(X_n)]$$
$$= c\theta^{-n} + [bn/(n+1)](\theta - \theta^{-n}).$$

Thus, $c = 1$ and $b = (n+1)/n$.

The UMVUE of $\theta$ is then

$$h(X_n) = \begin{cases} 
1 & 0 \leq X_n \leq 1 \\
(1 + n^{-1})X_n & X_n > 1.
\end{cases}$$

This estimator is better than $(1 + n^{-1})X_n$, which is the UMVUE when $\Theta = (0, \infty)$ and does not make use of the information about $\theta \geq 1$.

When $\Theta = (0, \infty)$, this estimator is not unbiased.

In fact, $h(X_n)$ is complete and sufficient for $\theta \in [1, \infty)$. 
Example 3.7 (continued)

From the previous discussion,

\[ E[h(X(n))U(X(n))] = 0, \quad \theta \geq 1. \]

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From the previous discussion,

\[ E[h(X_{(n)})U(X_{(n)})] = 0, \quad \theta \geq 1. \]

Since \( E[h(X_{(n)})] = \theta \), we obtain that

\[
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Thus, \( c = 1 \) and \( b = (n+1)/n \).

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When \( \Theta = (0, \infty) \), this estimator is not unbiased.

In fact, \( h(X(n)) \) is complete and sufficient for \( \theta \in [1, \infty) \).
Example 3.7 (continued)

It suffices to show that

\[ g(X_{(n)}) = \begin{cases} 
1 & 0 \leq X_{(n)} \leq 1 \\
X_{(n)} & X_{(n)} > 1.
\end{cases} \]

is complete and sufficient for \( \theta \in [1, \infty) \).

The sufficiency follows from the fact that the joint p.d.f. of \( X_1, \ldots, X_n \) is

\[ \frac{1}{\theta^n} l_{(0,\theta)}(X_{(n)}) = \frac{1}{\theta^n} l_{(0,\theta)}(g(X_{(n)})). \]

If \( E[f(g(X_{(n)}))] = 0 \) for all \( \theta > 1 \), then

\[ 0 = \int_0^\theta f(g(x))x^{n-1} \, dx = \int_0^1 f(1)x^{n-1} \, dx + \int_1^\theta f(x)x^{n-1} \, dx \]

for all \( \theta > 1 \).

Letting \( \theta \to 1 \) we obtain that \( f(1) = 0 \).

Then

\[ 0 = \int_1^\theta f(x)x^{n-1} \, dx \]

for all \( \theta > 1 \), which implies \( f(x) = 0 \) a.e. for \( x > 1 \).

Hence, \( g(X_{(n)}) \) is complete.
Example 3.8

Let $X$ be a sample (of size 1) from the uniform distribution $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$, $\theta \in \mathbb{R}$.

We now apply Theorem 3.2 to show that there is no UMVUE of $\vartheta = g(\theta)$ for any nonconstant function $g$.

Note that an unbiased estimator $U(X)$ of 0 must satisfy

$$\int_{\theta - \frac{1}{2}}^{\theta + \frac{1}{2}} U(x) \, dx = 0 \quad \text{for all } \theta \in \mathbb{R}.$$

Differentiating both sides of the previous equation and applying the result of differentiation of an integral lead to

$$U(x) = U(x + 1) \quad \text{a.e. } m,$$

where $m$ is the Lebesgue measure on $\mathbb{R}$.

If $T$ is a UMVUE of $g(\theta)$, then $T(X)U(X)$ is unbiased for 0 and, hence,

$$T(x)U(x) = T(x + 1)U(x + 1) \quad \text{a.e. } m,$$

where $U(X)$ is any unbiased estimator of 0.
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$$T(x)U(x) = T(x + 1)U(x + 1) \quad \text{a.e. } m,$$

where $U(X)$ is any unbiased estimator of 0.
Example 3.8 (continued)

Since this is true for all $U$,

\[
T(x) = T(x + 1) \quad \text{a.e. } m.
\]

Since $T$ is unbiased for $g(\theta)$,

\[
g(\theta) = \int_{\theta - \frac{1}{2}}^{\theta + \frac{1}{2}} T(x) \, dx \quad \text{for all } \theta \in \mathbb{R}.
\]

Differentiating both sides of the previous equation and applying the result of differentiation of an integral, we obtain that

\[
g'(\theta) = T \left( \theta + \frac{1}{2} \right) - T \left( \theta - \frac{1}{2} \right) = 0 \quad \text{a.e. } m.
\]