U-statistics

Let $X_1, \ldots, X_n$ be i.i.d. from an unknown population $P$ in a nonparametric family $\mathcal{P}$.

If the vector of order statistic is sufficient and complete for $P \in \mathcal{P}$, then a symmetric unbiased estimator of an estimable $\vartheta$ is the UMVUE of $\vartheta$.

In many problems, parameters to be estimated are of the form

$$\vartheta = E[h(X_1, \ldots, X_m)]$$

with a positive integer $m$ and a Borel function $h$ that is symmetric and satisfies

$$E|h(X_1, \ldots, X_m)| < \infty \quad \text{for any } P \in \mathcal{P}.$$ 

It is easy to see that a symmetric unbiased estimator of $\vartheta$ is

$$U_n = \left(\binom{n}{m}\right)^{-1} \sum_{c} h(X_{i_1}, \ldots, X_{i_m}),$$

where $\sum_c$ denotes the summation over the $\binom{n}{m}$ combinations of $m$ distinct elements $\{i_1, \ldots, i_m\}$ from $\{1, \ldots, n\}$. 
Definition 3.2

The statistic

\[ U_n = \left( \binom{n}{m} \right)^{-1} \sum_{c} h(X_{i_1}, ..., X_{i_m}), \]

is called a **U-statistic** with kernel \( h \) of order \( m \).

Remarks

- The use of U-statistics is an effective way of obtaining unbiased estimators.

- In nonparametric problems, U-statistics are often UMVUE’s, whereas in parametric problems, U-statistics can be used as initial estimators to derive more efficient estimators.

- If \( m = 1 \), a U-statistic is simply a type of sample mean. Examples include the empirical c.d.f. evaluated at a particular \( t \) and the **sample moments** \( n^{-1} \sum_{i=1}^{n} X_i^k \) for a positive integer \( k \).
Examples

Consider the estimation of $\vartheta = \mu^m$, where $\mu = \mathbb{E}X_1$ and $m$ is a positive integer. Using $h(x_1, \ldots, x_m) = x_1 \cdots x_m$, we obtain the following U-statistic unbiased for $\vartheta = \mu^m$:

$$U_n = \left( \begin{pmatrix} n \\ m \end{pmatrix} \right)^{-1} \sum_{c} X_{i_1} \cdots X_{i_m}.$$

Consider the estimation of $\vartheta = \sigma^2 = \text{Var}(X_1)$. Since

$$\sigma^2 = \left[ \text{Var}(X_1) + \text{Var}(X_2) \right]/2 = \mathbb{E}[(X_1 - X_2)^2/2],$$

we obtain the following U-statistic with kernel $h(x_1, x_2) = (x_1 - x_2)^2/2$:

$$U_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \frac{(X_i - X_j)^2}{2} = \frac{1}{n-1} \left( \sum_{i=1}^{n} X_i^2 - n\bar{X}^2 \right) = S^2,$$

which is the sample variance.
Examples

In some cases, we would like to estimate $\vartheta = E|X_1 - X_2|$, a measure of concentration.
Using kernel $h(x_1, x_2) = |x_1 - x_2|$, we obtain the following U-statistic unbiased for $\vartheta = E|X_1 - X_2|:$

$$U_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j|,$$

which is known as \textit{Gini’s mean difference}.

Let $\vartheta = P(X_1 + X_2 \leq 0)$.
Using kernel $h(x_1, x_2) = I_{(-\infty,0]}(x_1 + x_2)$, we obtain the following U-statistic unbiased for $\vartheta$:

$$U_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} I_{(-\infty,0]}(X_i + X_j),$$

which is known as the \textit{one-sample Wilcoxon statistic}. 

Variance of a U-statistic

If $E[h(X_1, ..., X_m)]^2 < \infty$, then the variance of a U-statistic $U_n$ with kernel $h$ has an explicit form.

To derive $\text{Var}(U_n)$, we need some notation.

**Notation**

For $k = 1, ..., m$, let

$$h_k(x_1, ..., x_k) = E[h(X_1, ..., X_m)|X_1 = x_1, ..., X_k = x_k]$$

$$= E[h(x_1, ..., x_k, X_{k+1}, ..., X_m)].$$

Note that $h_m = h$.

It can be shown that

$$h_k(x_1, ..., x_k) = E[h_{k+1}(x_1, ..., x_k, X_{k+1})].$$

Define

$$\tilde{h}_k = h_k - E[h(X_1, ..., X_m)],$$

$k = 1, ..., m$, and $\tilde{h} = \tilde{h}_m$. 
For any U-statistic

\[ U_n = \binom{n}{m}^{-1} \sum_c h(X_{i1}, \ldots, X_{im}), \]

we have

\[ U_n - E(U_n) = \binom{n}{m}^{-1} \sum_c \tilde{h}(X_{i1}, \ldots, X_{im}). \]  

(1)

**Theorem 3.4 (Hoeffding’s theorem)**

For a U-statistic \( U_n \) with \( E[h(X_1, \ldots, X_m)]^2 < \infty \),

\[ \text{Var}(U_n) = \binom{n}{m}^{-1} \sum_{k=1}^{m} \binom{m}{k} \binom{n-m}{m-k} \zeta_k, \]

where

\[ \zeta_k = \text{Var}(h_k(X_1, \ldots, X_k)). \]
Proof

Consider two sets \( \{i_1, \ldots, i_m\} \) and \( \{j_1, \ldots, j_m\} \) of \( m \) distinct integers from \( \{1, \ldots, n\} \) with exactly \( k \) integers in common. The number of distinct choices of two such sets is \( \binom{n}{m} \binom{m}{k} \binom{n-m}{m-k} \).

By the symmetry of \( \tilde{h}_m \) and independence of \( X_1, \ldots, X_n \),

\[
E[\tilde{h}(X_{i_1}, \ldots, X_{i_m})\tilde{h}(X_{j_1}, \ldots, X_{j_m})] = \zeta_k
\]

for \( k = 1, \ldots, m \).

Then, by (1),

\[
\text{Var}(U_n) = \binom{n}{m}^{-2} \sum_c \sum_c E[\tilde{h}(X_{i_1}, \ldots, X_{i_m})\tilde{h}(X_{j_1}, \ldots, X_{j_m})] = \binom{n}{m}^{-2} \sum_{k=1}^{m} \binom{n}{m} \binom{m}{k} \binom{n-m}{m-k} \zeta_k.
\]

This proves the result.
Corollary 3.2

Under the condition of Theorem 3.4,

(i) \( \frac{m^2}{n} \zeta_1 \leq \text{Var}(U_n) \leq \frac{m}{n} \zeta_m; \)

(ii) \( (n+1) \text{Var}(U_{n+1}) \leq n \text{Var}(U_n) \) for any \( n > m; \)

(iii) For any fixed \( m \) and \( k = 1, \ldots, m \), if \( \zeta_j = 0 \) for \( j < k \) and \( \zeta_k > 0 \), then

\[
\text{Var}(U_n) = \frac{k!(\binom{m}{k})^2 \zeta_k}{n^k} + O\left(\frac{1}{n^{k+1}}\right).
\]

Remarks

- It follows from Corollary 3.2 that a U-statistic \( U_n \) as an estimator of its mean is consistent in mse (under the finite second moment assumption on \( h \)).

- In fact, for any fixed \( m \), if \( \zeta_j = 0 \) for \( j < k \) and \( \zeta_k > 0 \), then the mse of \( U_n \) is of the order \( n^{-k} \) and, therefore, \( U_n \) is \( n^{k/2} \)-consistent.
Example 3.11

Consider first \( h(x_1, x_2) = x_1 x_2 \), which leads to a U-statistic unbiased for \( \mu^2 \), where \( \mu = E X_1 \).

Note that \( h_1(x_1) = \mu x_1 \), \( \tilde{h}_1(x_1) = \mu (x_1 - \mu) \), \( \zeta_1 = E[\tilde{h}_1(X_1)]^2 = \mu^2 \) \( \text{Var}(X_1) = \mu^2 \sigma^2 \), \( \tilde{h}(x_1, x_2) = x_1 x_2 - \mu^2 \), and \( \zeta_2 = \text{Var}(X_1 X_2) = E(X_1 X_2)^2 - \mu^4 = (\mu^2 + \sigma^2)^2 - \mu^4 \).

By Theorem 3.4, for

\[
U_n = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} X_i X_j,
\]

\[
\text{Var}(U_n) = \binom{n}{2}^{-1} \left[ \binom{2}{1} \binom{n-2}{1} \zeta_1 + \binom{2}{2} \binom{n-2}{0} \zeta_2 \right]
\]

\[
= \frac{2}{n(n-1)} \left[ 2(n-2)\mu^2 \sigma^2 + (\mu^2 + \sigma^2)^2 - \mu^4 \right]
\]

\[
= \frac{4\mu^2 \sigma^2}{n} + \frac{2\sigma^4}{n(n-1)}.
\]
Example 3.11 (continued)

Comparing $U_n$ with $\bar{X}^2 - \sigma^2 / n$ in Example 3.10, which is the UMVUE under the normality and known $\sigma^2$ assumption, we find that

$$\text{Var}(U_n) - \text{Var}(\bar{X}^2 - \sigma^2 / n) = \frac{2\sigma^4}{n^2(n-1)}.$$ 

Next, consider $h(x_1, x_2) = I_{(-\infty, 0]}(x_1 + x_2)$, which leads to the one-sample Wilcoxon statistic.

Note that $h_1(x_1) = P(x_1 + X_2 \leq 0) = F(-x_1)$, where $F$ is the c.d.f. of $P$.

Then $\zeta_1 = \text{Var}(F(-X_1))$.

Let $\vartheta = E[h(X_1, X_2)]$.

Then $\zeta_2 = \text{Var}(h(X_1, X_2)) = \vartheta(1 - \vartheta)$.

Hence, for $U_n$ being the one-sample Wilcoxon statistic,

$$\text{Var}(U_n) = \frac{2}{n(n-1)} \left[ 2(n-2)\zeta_1 + \vartheta(1 - \vartheta) \right].$$

If $F$ is continuous and symmetric about 0, then $\zeta_1$ can be simplified as

$$\zeta_1 = \text{Var}(F(-X_1)) = \text{Var}(1 - F(X_1)) = \text{Var}(F(X_1)) = \frac{1}{12}.$$
V-statistics

Let $X_1, \ldots, X_n$ be i.i.d. from $P$.

For every U-statistic $U_n$ as an estimator of $\vartheta = E[h(X_1, \ldots, X_m)]$, there is a closely related V-statistic defined by

$$ V_n = \frac{1}{n^m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} h(X_{i_1}, \ldots, X_{i_m}). $$

(2)

As an estimator of $\vartheta$, $V_n$ is biased; but the bias is small asymptotically as the following results show.

For a fixed sample size $n$, $V_n$ may be better than $U_n$ in terms of their mse’s.

Example

Corresponding to $U_n = \frac{2}{n(n-1)} \sum_{i<j} X_iX_j$, the V-statistic is

$$ V_n = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} X_iX_j = (\bar{X})^2 $$
Proposition 3.5

Let $V_n$ be defined by (2).

(i) Assume that $E|h(X_{i_1}, \ldots, X_{i_m})| < \infty$ for all $1 \leq i_1 \leq \cdots \leq i_m \leq m$. Then the bias of $V_n$ satisfies

$$b_{V_n}(P) = O(n^{-1}).$$

(ii) Assume that $E[h(X_{i_1}, \ldots, X_{i_m})]^2 < \infty$ for all $1 \leq i_1 \leq \cdots \leq i_m \leq m$. Then the variance of $V_n$ satisfies

$$\text{Var}(V_n) = \text{Var}(U_n) + O(n^{-2}),$$

where $U_n$ is the U-statistic corresponding to $V_n$. 