Lecture 20: Inverting acceptance regions of tests, UMA and UMAU confidence sets

Confidence sets and hypothesis tests

Another popular method of constructing confidence sets is to use a close relationship between confidence sets and hypothesis tests.
For any test T, the set $\{x : T(x) \neq 1\}$ is called the *acceptance region*.
This terminology is not precise when T is a randomized test.

Theorem 7.2

For each $\theta_0 \in \Theta$, let T_{θ_0} be a test for $H_0 : \theta = \theta_0$ (versus some H_1) with significance level α and acceptance region $A(\theta_0)$.
For each x in the range of X, define

$$C(x) = \{\theta : x \in A(\theta)\}.$$

Then $C(X)$ is a level $1-\alpha$ confidence set for θ.
If T_{θ_0} is nonrandomized and has size α for every θ_0, then $C(X)$ has confidence coefficient $1-\alpha$.

Proof

We prove the first assertion only.
The proof for the second assertion is similar.
Under the given condition,

$$\sup_{\theta=\theta_0} P\big(X \notin A(\theta_0)\big) = \sup_{\theta=\theta_0} P(T_{\theta_0} = 1) \leq \alpha,$$

which is the same as

$$1-\alpha \leq \inf_{\theta=\theta_0} P\big(X \in A(\theta_0)\big) = \inf_{\theta=\theta_0} P\big(\theta_0 \in C(X)\big).$$

Since this holds for all θ_0, the result follows from

$$\inf_{P\in\mathscr{P}} P\big(\theta \in C(X)\big) = \inf_{\theta_0\in\Theta}\inf_{\theta=\theta_0} P\big(\theta_0 \in C(X)\big) \geq 1-\alpha.$$

The converse of Theorem 7.2 is partially true.

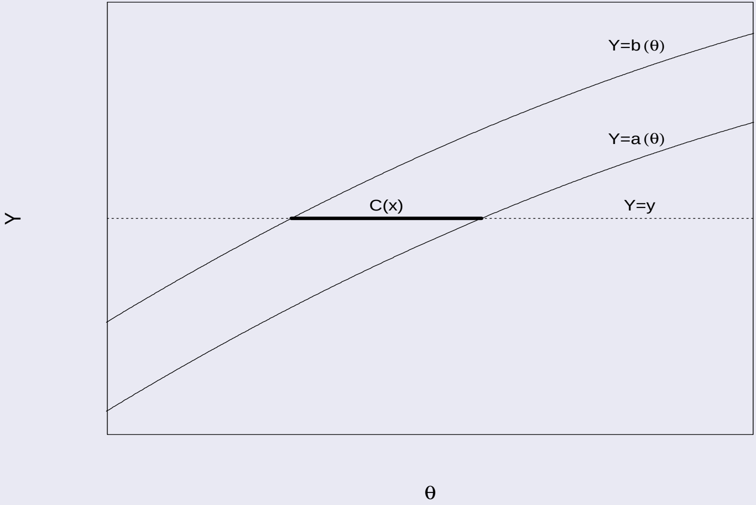
Proposition 7.2

Let $C(X)$ be a confidence set for θ with confidence level (or confidence coefficient) $1-\alpha$.
For any $\theta_0 \in \Theta$, define a region $A(\theta_0) = \{x : \theta_0 \in C(x)\}$.
Then the test $T(X) = 1 - I_{A(\theta_0)}(X)$ has significance level α for testing $H_0 : \theta = \theta_0$ versus some H_1.

Discussions

In general, $C(X)$ in Theorem 7.2 can be determined numerically, if it does not have an explicit form.
Suppose $A(\theta) = \{Y : a(\theta) \le Y \le b(\theta)\}$ for a real-valued θ and statistic $Y(X)$ and some nondecreasing functions $a(\theta)$ and $b(\theta)$.
When we observe $Y = y$, $C(X)$ is an interval with limits $\underline{\theta}$ and $\overline{\theta}$, which are the θ-values at which the horizontal line $Y = y$ intersects the curves $Y = b(\theta)$ and $Y = a(\theta)$ (Figure 7.1), respectively.
If $y = b(\theta)$ (or $y = a(\theta)$) has no solution or more than one solution, $\underline{\theta} = \inf\{\theta : y \le b(\theta)\}$ (or $\overline{\theta} = \sup\{\theta : a(\theta) \le y\}$).
$C(X)$ does not include $\underline{\theta}$ (or $\overline{\theta}$) if and only if at $\underline{\theta}$ (or $\overline{\theta}$), $b(\theta)$ (or $a(\theta)$) is only left-continuous (or right-continuous).

Figure 7.1. A confidence interval obtained by inverting $A(\theta) = [a(\theta), b(\theta)]$



Example 7.7

Suppose that X has the following p.d.f. in a one-parameter exponential family:

$$f_\theta(x) = \exp\{\eta(\theta)Y(x) - \xi(\theta)\}h(x),$$

where θ is real-valued and $\eta(\theta)$ is nondecreasing in θ.
First, we apply Theorem 7.2 with $H_0 : \theta = \theta_0$ and $H_1 : \theta > \theta_0$.
By Theorem 6.2, the acceptance region of the UMP test of size α is

$$A(\theta_0) = \{x : Y(x) \le c(\theta_0)\},$$

where $c(\theta_0) = c$ in Theorem 6.2.
It can be shown that $c(\theta)$ is nondecreasing in θ.
Inverting $A(\theta)$ according to Figure 7.1 with $b(\theta) = c(\theta)$ and $a(\theta)$ ignored, we obtain

$$C(X) = [\underline{\theta}(X), \infty) \quad \text{or} \quad (\underline{\theta}(X), \infty),$$

a one-sided confidence interval for θ with confidence level $1-\alpha$.
$\underline{\theta}(X)$ is a called a lower confidence bound for θ in §2.4.3.
When the c.d.f. of $Y(X)$ is continuous, $C(X)$ has confidence coefficient $1-\alpha$.

If $H_0 : \theta = \theta_0$ and $H_1 : \theta < \theta_0$ are considered, then $C(X) = \{\theta : Y(X) \geq c(\theta)\}$ and is of the form

$$(-\infty, \overline{\theta}(X)] \quad \text{or} \quad (-\infty, \overline{\theta}(X)).$$

$\overline{\theta}(X)$ is called an upper confidence bound for θ.

Consider next $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$.
By Theorem 6.4, the acceptance region of the UMPU test of size α is given by $A(\theta_0) = \{x : c_1(\theta_0) \leq Y(x) \leq c_2(\theta_0)\}$, where $c_i(\theta)$ are nondecreasing (exercise).
A confidence interval can be obtained by inverting $A(\theta)$ according to Figure 7.1 with $a(\theta) = c_1(\theta)$ and $b(\theta) = c_2(\theta)$.

Let us consider a specific example in which $X_1, ..., X_n$ are i.i.d. binary random variables with $p = P(X_i = 1)$.
Note that $Y(X) = \sum_{i=1}^n X_i$.
Suppose that we need a lower confidence bound for p so that we consider $H_0 : p = p_0$ and $H_1 : p > p_0$.

From Example 6.2, the acceptance region of a UMP test of size $\alpha \in (0,1)$ is $A(p_0) = \{y : y \leq m(p_0)\}$, where $m(p_0)$ is an integer between 0 and n such that

$$\sum_{j=m(p_0)+1}^{n} \binom{n}{j} p_0^j (1-p_0)^{n-j} \leq \alpha < \sum_{j=m(p_0)}^{n} \binom{n}{j} p_0^j (1-p_0)^{n-j}.$$

Thus, $m(p)$ is an integer-valued, left-continuous, nondecreasing step-function of p.
Define

$$\underline{p} = \inf\{p : m(p) \geq y\} = \inf\left\{p : \sum_{j=y}^{n} \binom{n}{j} p^j (1-p)^{n-j} > \alpha\right\}.$$

We want to show that a level $1-\alpha$ confidence interval for p is $(\underline{p}, 1]$.
Inverting $A(p)$ we obtain that

$$C(y) = \{p : y \leq m(p)\}$$

We need to show that

$$\{p : y \leq m(p)\} = \{p : \underline{p} < p\}$$

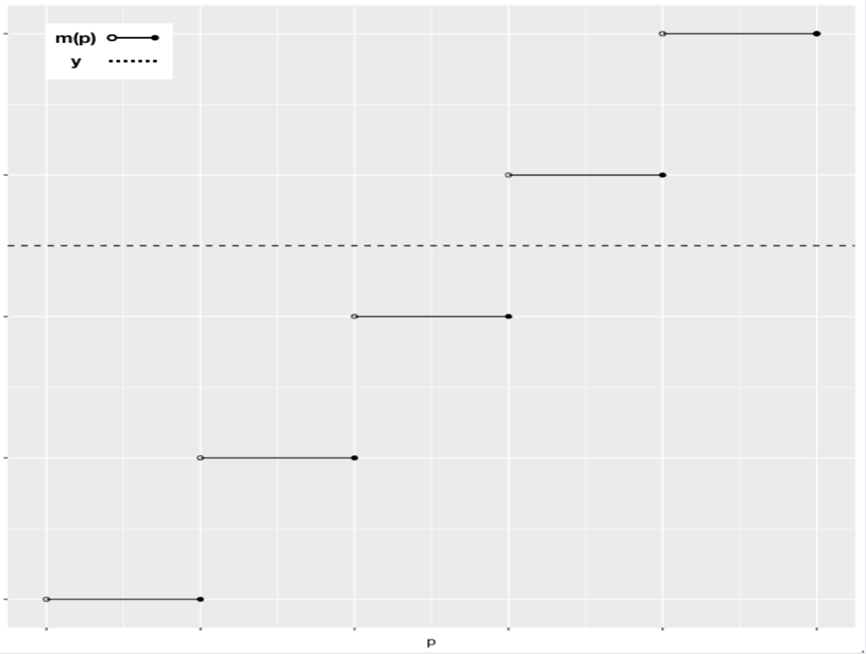
Suppose that $\underline{p} < p$.
If $m(p) < y$, then, by the definition of $\underline{p}$, we must have $p \leq \underline{p}$, a contradiction.
Hence, we must have $y \leq m(p)$.
This shows

$$\{p : \underline{p} < p\} \subset \{p : y \leq m(p)\}$$

Suppose that $y \leq m(p)$.
By the definition of $\underline{p}$, $\underline{p} \leq p$.
But we cannot have $\underline{p} = p$, because $m(p)$ is left-continuous and flat, i.e., if $y \leq m(\underline{p})$, then there is a $p < \underline{p}$ such that $y \leq m(p)$.
Thus, $\underline{p} < p$ and, hence,

$$\{p : y \leq m(p)\} \subset \{p : \underline{p} < p\}$$

One can compare this confidence interval with the one obtained by applying Theorem 7.1 (exercise).
See also Example 7.16.


m(p)
y
p

Example 7.8

Suppose that X has the following p.d.f. in a multiparameter exponential family:

$$f_{\theta,\varphi}(x) = \exp\{\theta Y(x) + \varphi^\tau U(x) - \zeta(\theta,\varphi)\}$$

By Theorem 6.4, the acceptance region of a UMPU test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ or $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is

$$A(\theta_0) = \{(y,u) : y \leq c_2(u,\theta_0)\}$$

or

$$A(\theta_0) = \{(y,u) : c_1(u,\theta_0) \leq y \leq c_2(u,\theta_0)\},$$

where $c_i(u,\theta)$, $i = 1,2$, are nondecreasing functions of θ.
Confidence intervals for θ can then be obtained by inverting $A(\theta)$ according to Figure 7.1 with $b(\theta) = c_2(u,\theta)$ and $a(\theta) = c_1(u,\theta)$ or $a(\theta) \equiv -\infty$, for any observed u.

Consider more specifically the case where X_1 and X_2 are independently distributed as the Poisson distributions $P(\lambda_1)$ and $P(\lambda_2)$, respectively, and we need a lower confidence bound for the ratio $\rho = \lambda_2/\lambda_1$.

From Example 6.11, a UMPU test of size α for testing $H_0 : \rho = \rho_0$ versus $H_1 : \rho > \rho_0$ has the acceptance region

$$A(\rho_0) = \{(y,u) : y \le c(u,\rho_0)\},$$

where $c(u,\rho_0)$ is determined by the conditional distribution of $Y = X_2$ given $U = X_1 + X_2 = u$.
Since the conditional distribution of Y given $U = u$ is the binomial distribution $Bi(\rho/(1+\rho), u)$, we can use the result in Example 7.7, i.e., $c(u,\rho)$ is the same as $m(p)$ in Example 7.7 with $n = u$ and $p = \rho/(1+\rho)$.
Then a level $1-\alpha$ lower confidence bound for p is $\underline{p}$ given by

$$\underline{p} = \inf\{p : m(p) \ge y\} = \inf\left\{ p : \sum_{j=y}^{u} \binom{u}{j} p^j (1-p)^{u-j} \ge \alpha \right\}$$

Since $\rho = p/(1-p)$ is a strictly increasing function of p, a level $1-\alpha$ lower confidence bound for ρ is $\underline{p}/(1-\underline{p})$.

Confidence sets related to optimal tests

For a confidence set obtained by inverting the acceptance regions of some UMP or UMPU tests, it is expected that the confidence set inherits some optimality property.

Definition 7.2

Let $\theta \in \Theta$ be an unknown parameter and Θ' be a subset of Θ that does not contain the true parameter value θ.
A confidence set $C(X)$ for θ with confidence coefficient $1 - \alpha$ is said to be Θ'-*uniformly most accurate* (UMA) iff for any other confidence set $C_1(X)$ with confidence level $1 - \alpha$,

$$P\big(\theta' \in C(X)\big) \leq P\big(\theta' \in C_1(X)\big) \qquad \text{for all } \theta' \in \Theta'.$$

$C(X)$ is UMA iff it is Θ'-UMA with $\Theta' = \{\theta\}^c$.

- Intuitively, confidence sets with small probabilities of covering wrong parameter values are preferred.
- If we consider a lower confidence bound for a real-valued θ, we only need to worry about covering values of θ that are too small, i.e., $\Theta' = \{\theta' \in \Theta : \theta' < \theta\}$.

Theorem 7.4

Let $C(X)$ be a confidence set for θ obtained by inverting the acceptance regions of nonrandomized tests T_{θ_0} for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_{\theta_0}$.
Suppose that for each θ_0, T_{θ_0} is UMP of size α.
Then $C(X)$ is Θ'-UMA with confidence coefficient $1 - \alpha$, where $\Theta' = \{\theta' : \theta \in \Theta_{\theta'}\}$.

Proof

The fact that $C(X)$ has confidence coefficient $1 - \alpha$ follows from Theorem 7.2.
Let $C_1(X)$ be another confidence set with confidence level $1 - \alpha$.
By Proposition 7.2, the test

$$T_{1\theta_0}(X) = 1 - I_{A_1(\theta_0)}(X)$$

with $A_1(\theta_0) = \{x : \theta_0 \in C_1(x)\}$ has significance level α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_{\theta_0}$.

For any $\theta' \in \Theta'$, $\theta \in \Theta_{\theta'}$, i.e., P is in the family defined by $H_1 : \theta \in \Theta_{\theta'}$. Thus,

$$\begin{aligned} P\big(\theta' \in C(X)\big) &= 1 - P\big(T_{\theta'}(X) = 1\big) \\ &\leq 1 - P\big(T_{1\theta'}(X) = 1\big) \\ &= P\big(\theta' \in C_1(X)\big), \end{aligned}$$

where the first equality follows from the fact that $T_{\theta'}$ is nonrandomized and the inequality follows from the fact that $T_{\theta'}$ is UMP.

Discussions

Theorem 7.4 can be applied to construct UMA confidence bounds in problems where the population is in a one-parameter parametric family with monotone likelihood ratio so that UMP tests exist (Theorem 6.2).

It can also be applied to a few cases to construct two-sided UMA confidence intervals.

For example, $[X_{(n)}, \alpha^{-1/n} X_{(n)}]$ in Example 7.13 is UMA.

As we discussed in §6.2, in many problems there are UMPU tests but not UMP tests.

Definition 7.3

Let $\theta \in \Theta$ be an unknown parameter, Θ' be a subset of Θ that does not contain the true parameter value θ, and $1 - \alpha$ be a given confidence level.

(i) A level $1 - \alpha$ confidence set $C(X)$ is said to be Θ'-*unbiased* (unbiased when $\Theta' = \{\theta\}^c$) iff

$$P(\theta' \in C(X)) \leq 1 - \alpha$$

for all $\theta' \in \Theta'$.

(ii) Let $C(X)$ be a Θ'-unbiased confidence set with confidence coefficient $1 - \alpha$. If

$$P\big(\theta' \in C(X)\big) \leq P\big(\theta' \in C_1(X)\big) \qquad \text{for all } \theta' \in \Theta'.$$

holds for any other Θ'-unbiased confidence set $C_1(X)$ with confidence level $1 - \alpha$, then $C(X)$ is Θ'-*uniformly most accurate unbiased* (UMAU). $C(X)$ is UMAU if and only if it is Θ'-UMAU with $\Theta' = \{\theta\}^c$.

Theorem 7.5

Let $C(X)$ be a confidence set for θ obtained by inverting the acceptance regions of nonrandomized tests T_{θ_0} for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_{\theta_0}$.
If T_{θ_0} is unbiased of size α for each θ_0, then $C(X)$ is Θ'-unbiased with confidence coefficient $1-\alpha$, where $\Theta' = \{\theta' : \theta \in \Theta_{\theta'}\}$.
If T_{θ_0} is also UMPU for each θ_0, then $C(X)$ is Θ'-UMAU.

Examples 7.9 and 7.15.

Consider the normal linear model $X = N_n(Z\beta, \sigma^2 I_n)$ and the problem of constructing a confidence set for $\theta = L\beta$, where L is an $s \times p$ matrix of rank s and all rows of L are in $\mathscr{R}(Z)$.
The LR test of size α for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has the acceptance region

$$A(\theta_0) = \{X : W(X, \theta_0) \leq c_\alpha\},$$

where c_α is the $(1-\alpha)$th quantile of the F-distribution $F_{s,n-r}$,

$$W(X,\theta) = \frac{[\|X - Z\widehat{\beta}(\theta)\|^2 - \|X - Z\widehat{\beta}\|^2]/s}{\|X - Z\widehat{\beta}\|^2/(n-r)},$$

r is the rank of Z, $r \geq s$, $\widehat{\beta}$ is the LSE of β and, for each fixed θ, $\widehat{\beta}(\theta)$ is a solution of

$$\|X - Z\widehat{\beta}(\theta)\|^2 = \min_{\beta : L\beta = \theta} \|X - Z\beta\|^2.$$

Inverting $A(\theta)$, we obtain the following confidence set for θ with confidence coefficient $1 - \alpha$: $C(X) = \{\theta : W(X, \theta) \leq c_\alpha\}$, which forms a closed ellipsoid in $\mathscr{R}^s$.

Consider the special case of $s = 1$, $\theta = l^\tau \beta$, where $l \in \mathscr{R}(Z)$.
From §6.2.3, the nonrandomized test with acceptance region

$$A(\theta_0) = \left\{ X : l^\tau \widehat{\beta} - \theta_0 > t_{n-r,\alpha} \sqrt{l^\tau (Z^\tau Z)^- l SSR/(n-r)} \right\}$$

is UMPU with size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$, where $t_{n-r,\alpha}$ is the $(1-\alpha)$th quantile of the t-distribution t_{n-r}.
Inverting $A(\theta)$ we obtain the following Θ'-UMAU upper confidence bound with confidence coefficient $1 - \alpha$ and $\Theta' = (\theta, \infty)$:

$$\overline{\theta} = l^\tau \widehat{\beta} - t_{n-r,\alpha} \sqrt{l^\tau (Z^\tau Z)^- l SSR/(n-r)}.$$

A UMAU confidence interval for θ can be similarly obtained.