Continuity of the power function

For a given test $T$, the power function $\beta_T(P)$ is said to be continuous in $\theta$ if and only if for any $\{\theta_j : j = 0, 1, 2, \ldots\} \subset \Theta$, $\theta_j \to \theta_0$ implies $\beta_T(P_j) \to \beta_T(P_0)$, where $P_j \in \mathcal{P}$ satisfying $\theta(P_j) = \theta_j$, $j = 0, 1, \ldots$. If $\beta_T$ is a function of $\theta$, then this continuity property is simply the continuity of $\beta_T(\theta)$.

Lemma 6.5

Consider hypotheses $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$. Suppose that, for every $T$, $\beta_T(P)$ is continuous in $\theta$. If $T_*$ is uniformly most powerful among all similar tests and has size $\alpha$, then $T_*$ is a UMPU test.

Proof

Under the continuity assumption on $\beta_T$, the class of similar tests contains the class of unbiased tests. Since $T_*$ is uniformly at least as powerful as the test $T \equiv \alpha$, $T_*$ is unbiased.
Continuity of the power function

For a given test $T$, the power function $\beta_T(P)$ is said to be continuous in $\theta$ if and only if for any $\{\theta_j : j = 0, 1, 2, \ldots\} \subset \Theta$, $\theta_j \to \theta_0$ implies $\beta_T(P_j) \to \beta_T(P_0)$, where $P_j \in \mathcal{P}$ satisfying $\theta(P_j) = \theta_j$, $j = 0, 1, \ldots$.

If $\beta_T$ is a function of $\theta$, then this continuity property is simply the continuity of $\beta_T(\theta)$.

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Consider hypotheses $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$.

Suppose that, for every $T$, $\beta_T(P)$ is continuous in $\theta$.

If $T_*$ is uniformly most powerful among all similar tests and has size $\alpha$, then $T_*$ is a UMPU test.

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Under the continuity assumption on $\beta_T$, the class of similar tests contains the class of unbiased tests.

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Continuity of the power function

For a given test $T$, the power function $\beta_T(P)$ is said to be continuous in $\theta$ if and only if for any $\{\theta_j : j = 0, 1, 2, \ldots\} \subset \Theta$, $\theta_j \to \theta_0$ implies $\beta_T(P_j) \to \beta_T(P_0)$, where $P_j \in \mathcal{P}$ satisfying $\theta(P_j) = \theta_j$, $j = 0, 1, \ldots$. If $\beta_T$ is a function of $\theta$, then this continuity property is simply the continuity of $\beta_T(\theta)$.

Lemma 6.5

Consider hypotheses $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$. Suppose that, for every $T$, $\beta_T(P)$ is continuous in $\theta$. If $T^*$ is uniformly most powerful among all similar tests and has size $\alpha$, then $T^*$ is a UMPU test.

Proof

Under the continuity assumption on $\beta_T$, the class of similar tests contains the class of unbiased tests. Since $T^*$ is uniformly at least as powerful as the test $T \equiv \alpha$, $T^*$ is unbiased.
Neyman structure

Let $U(X)$ be a sufficient statistic for $P \in \bar{\mathcal{P}}$ and let $\bar{\mathcal{P}}_U$ be the family of distributions of $U$ as $P$ ranges over $\bar{\mathcal{P}}$.

A test is said to have *Neyman structure* w.r.t. $U$ if

$$E[T(X)|U] = \alpha \quad \text{a.s. } \bar{\mathcal{P}}_U,$$

Clearly, if $T$ has Neyman structure, then

$$E[T(X)] = E\{E[T(X)|U]\} = \alpha \quad P \in \bar{\mathcal{P}},$$

i.e., $T$ is similar on $\bar{\Theta}_{01}$.

If all tests similar on $\bar{\Theta}_{01}$ have Neyman structure w.r.t. $U$, then working with tests having Neyman structure is the same as working with tests similar on $\bar{\Theta}_{01}$.

Lemma 6.6

Let $U(X)$ be a sufficient statistic for $P \in \bar{\mathcal{P}}$.

A necessary and sufficient condition for all tests similar on $\bar{\Theta}_{01}$ to have Neyman structure w.r.t. $U$ is that $U$ is boundedly complete for $P \in \bar{\mathcal{P}}$. 
Neyman structure

Let $U(X)$ be a sufficient statistic for $P \in \tilde{\mathcal{P}}$ and let $\tilde{\mathcal{P}}_U$ be the family of distributions of $U$ as $P$ ranges over $\tilde{\mathcal{P}}$. A test is said to have Neyman structure w.r.t. $U$ if

$$E[T(X)|U] = \alpha \quad \text{a.s. } \tilde{\mathcal{P}}_U,$$

Clearly, if $T$ has Neyman structure, then

$$E[T(X)] = E\{E[T(X)|U]\} = \alpha \quad P \in \tilde{\mathcal{P}},$$

i.e., $T$ is similar on $\tilde{\Theta}_{01}$. If all tests similar on $\tilde{\Theta}_{01}$ have Neyman structure w.r.t. $U$, then working with tests having Neyman structure is the same as working with tests similar on $\tilde{\Theta}_{01}$.

Lemma 6.6

Let $U(X)$ be a sufficient statistic for $P \in \tilde{\mathcal{P}}$. A necessary and sufficient condition for all tests similar on $\tilde{\Theta}_{01}$ to have Neyman structure w.r.t. $U$ is that $U$ is boundedly complete for $P \in \tilde{\mathcal{P}}$. 
(i) Suppose first that $U$ is boundedly complete for $P \in \bar{P}$. Let $T(X)$ be a test similar on $\bar{\Theta}_{01}$. Then $E[T(X) - \alpha] = 0$ for all $P \in \bar{P}$. From the boundedness of $T(X)$, $E[T(X)|U]$ is bounded. Since $E\{E[T(X)|U] - \alpha\} = E[T(X) - \alpha] = 0$ for all $P \in \bar{P}$ and $U$ is boundedly complete, $E[T(X)|U] = \alpha$ a.s. $\bar{P}_U$, i.e., $T$ has Neyman structure.

(ii) Suppose now that all tests similar on $\bar{\Theta}_{01}$ have Neyman structure w.r.t. $U$.

Suppose also that $U$ is not boundedly complete for $P \in \bar{P}$. Then there is a function $h$ such that $|h(u)| \leq C$, $E[h(U)] = 0$ for all $P \in \bar{P}$, and $h(U) \neq 0$ with positive probability for some $P \in \bar{P}$. Let $T(X) = \alpha + ch(U)$, where $c = \min\{\alpha, 1 - \alpha\}/C$. Then $T$ is a test similar on $\bar{\Theta}_{01}$ but $T$ does not have Neyman structure w.r.t. $U$ (because $h(U) \neq 0$). Thus, $U$ must be boundedly complete for $P \in \bar{P}$. This proves the result.
Theorem 6.4 (UMPU tests in multiparameter exponential families)

Suppose that $X$ has the following p.d.f. w.r.t. a $\sigma$-finite measure:

$$f_{\theta, \varphi}(x) = \exp \{\theta Y(x) + \varphi^\tau U(x) - \zeta(\theta, \varphi)\},$$

where $\theta$ is a real-valued parameter, $\varphi$ is a vector-valued parameter, and $Y$ (real-valued) and $U$ (vector-valued) are statistics.

(i) For testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, a UMPU test of size $\alpha$ is

$$T_*(Y, U) = \begin{cases} 
1 & Y > c(U) \\
\gamma(U) & Y = c(U) \\
0 & Y < c(U),
\end{cases}$$

where $c(u)$ and $\gamma(u)$ are Borel functions determined by

$$E_{\theta_0} [T_*(Y, U) | U = u] = \alpha \quad \text{for every } u$$

and $E_{\theta_0}$ is the expectation w.r.t. $f_{\theta_0, \varphi}$.

(ii) For testing $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ versus $H_1 : \theta_1 < \theta < \theta_2$, a UMPU test of size $\alpha$ is

$$T_*(Y, U) = \begin{cases} 
1 & c_1(U) < Y < c_2(U) \\
\gamma_i(U) & Y = c_i(U), \ i = 1, 2, \\
0 & Y < c_1(U) \text{ or } Y > c_2(U),
\end{cases}$$
Theorem 6.4 (continued)

where \( c_i(u) \)'s and \( \gamma_i(u) \)'s are Borel functions determined by

\[
E_{\theta_1}[T_*(Y, U)|U = u] = E_{\theta_2}[T_*(Y, U)|U = u] = \alpha \text{ for every } u.
\]

(iii) For testing \( H_0 : \theta_1 \leq \theta \leq \theta_2 \) versus \( H_1 : \theta < \theta_1 \) or \( \theta > \theta_2 \), a UMPU test of size \( \alpha \) is

\[
T_*(Y, U) = \begin{cases} 
1 & Y < c_1(U) \text{ or } Y > c_2(U) \\
\gamma_i(U) & Y = c_i(U), \ i = 1, 2, \\
0 & c_1(U) < Y < c_2(U),
\end{cases}
\]

where \( c_i(u) \)'s and \( \gamma_i(u) \)'s are Borel functions determined by

\[
E_{\theta_1}[T_*(Y, U)|U = u] = E_{\theta_2}[T_*(Y, U)|U = u] = \alpha \text{ for every } u.
\]

(iv) For testing \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta \neq \theta_0 \), a UMPU test of size \( \alpha \) is given by \( T_*(Y, U) \) in (iii), where \( c_i(u) \)'s and \( \gamma_i(u) \)'s are Borel functions determined by

\[
E_{\theta_0}[T_*(Y, U)|U = u] = \alpha \text{ for every } u
\]

and

\[
E_{\theta_0}[T_*(Y, U)Y|U = u] = \alpha E_{\theta_0}(Y|U = u) \text{ for every } u.
\]
Proof
By sufficiency, we only need to consider tests that are functions of \((Y, U)\).
It follows from Theorem 2.1(i) that the p.d.f. of \((Y, U)\) (w.r.t. a \(\sigma\)-finite measure) is in a natural exponential family of the form
\[
\exp \left\{ \theta y + \phi^\tau u - \zeta(\theta, \varphi) \right\}
\]
and, given \(U = u\), the p.d.f. of the conditional distribution of \(Y\) (w.r.t. a \(\sigma\)-finite measure \(\nu_u\)) is in a natural exponential family of the form
\[
\exp \left\{ \theta y - \zeta_u(\theta) \right\}.
\]
Hypotheses in (i)-(iv) are of the form \(H_0 : \theta \in \Theta_0\) vs \(H_1 : \theta \in \Theta_1\) with
\[
\bar{\Theta}_{01} = \{(\theta, \varphi) : \theta = \theta_0\} \text{ or } \{(\theta, \varphi) : \theta = \theta_i, \ i = 1, 2\}.
\]
In case (i) or (iv), \(U\) is sufficient and complete for \(P \in \bar{\mathcal{P}}\) and, hence, Lemma 6.6 applies.
In case (ii) or (iii), applying Lemma 6.6 to each \(\{(\theta, \varphi) : \theta = \theta_i\}\) also shows that working with tests having Neyman structure is the same as working with tests similar on \(\bar{\Theta}_{01}\).
By Theorem 2.1, the power functions of all tests are continuous and, hence, Lemma 6.5 applies.
Proof

By sufficiency, we only need to consider tests that are functions of $(Y, U)$.

It follows from Theorem 2.1(i) that the p.d.f. of $(Y, U)$ (w.r.t. a $\sigma$-finite measure) is in a natural exponential family of the form
\[
\exp \left\{ \theta y + \varphi^\tau u - \zeta(\theta, \varphi) \right\}
\]
and, given $U = u$, the p.d.f. of the conditional distribution of $Y$ (w.r.t. a $\sigma$-finite measure $\nu_u$) is in a natural exponential family of the form
\[
\exp \left\{ \theta y - \zeta_u(\theta) \right\}.
\]

Hypotheses in (i)-(iv) are of the form $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$ with
\[
\overline{\Theta}_{01} = \{ (\theta, \varphi) : \theta = \theta_0 \} \text{ or } \{ (\theta, \varphi) : \theta = \theta_i, i = 1, 2 \}.
\]

In case (i) or (iv), $U$ is sufficient and complete for $P \in \hat{P}$ and, hence, Lemma 6.6 applies.

In case (ii) or (iii), applying Lemma 6.6 to each $\{ (\theta, \varphi) : \theta = \theta_i \}$ also shows that working with tests having Neyman structure is the same as working with tests similar on $\overline{\Theta}_{01}$.

By Theorem 2.1, the power functions of all tests are continuous and, hence, Lemma 6.5 applies.
Thus, for (i), it suffices to show \( T_* \) is UMP among all tests \( T \) satisfying
\[
E_{\theta_0}[T(Y, U)|U = u] = \alpha \quad \text{for every } u \quad (1)
\]
and for part (ii) or (iii)), it suffices show \( T_* \) is UMP among all tests \( T \) satisfying
\[
E_{\theta_1}[T(Y, U)|U = u] = E_{\theta_2}[T(Y, U)|U = u] = \alpha \quad \text{for every } u.
\]
For (iv), any unbiased \( T \) should satisfy (1) and
\[
\frac{\partial}{\partial \theta} E_{\theta,\varphi}[T(Y, U)] = 0, \quad \theta \in \bar{\Theta}_{01}. \quad (2)
\]
One can show (exercise) that (2) is equivalent to
\[
E_{\theta,\varphi}[T(Y, U)Y - \alpha Y] = 0, \quad \theta \in \bar{\Theta}_{01}. \quad (3)
\]
Using the argument in the proof of Lemma 6.6, one can show (exercise) that (3) is equivalent to
\[
E_{\theta_0}[T(Y, U)Y|U = u] = \alpha E_{\theta_0}(Y|U = u) \quad \text{for every } u. \quad (4)
\]
Hence, for (iv), it suffices to show \( T_* \) is UMP among all tests \( T \) satisfying (1) and (4).
Note that the power function of any test $T(Y, U)$ is

$$\beta_T(\theta, \varphi) = \int \left[ \int T(y, u) dP_{Y|U=u}(y) \right] dP_U(u).$$

Thus, it suffices to show that for every fixed $u$ and $\theta \in \Theta_1$, $T^*$ maximizes

$$\int T(y, u) dP_{Y|U=u}(y)$$

over all $T$ subject to the given side conditions.

Since $P_{Y|U=u}$ is in a one-parameter exponential family, the results in (i) and (ii) follow from Corollary 6.1 and Theorem 6.3, respectively. The result in (iii) follows from Theorem 6.3(ii) by considering $1 - T^*$.

To prove the result in (iv), it suffices to show that if $Y$ has the p.d.f. given by $\exp \{ \theta y - \zeta_u(\theta) \}$ and if $u$ is treated as a constant in (1) and (4), $T^*$ in (iii) with a fixed $u$ is UMP subject to conditions (1) and (4). We now omit $u$ in the following proof for (iv), which is very similar to the proof of Theorem 6.3.
Note that the power function of any test \( T(Y, U) \) is

\[
\beta_T(\theta, \varphi) = \int \left[ \int T(y, u) dP_{Y|U=u}(y) \right] dP_U(u).
\]

Thus, it suffices to show that for every fixed \( u \) and \( \theta \in \Theta_1 \), \( T_\ast \) maximizes

\[
\int T(y, u) dP_{Y|U=u}(y)
\]

over all \( T \) subject to the given side conditions.
Since \( P_{Y|U=u} \) is in a one-parameter exponential family, the results in (i) and (ii) follow from Corollary 6.1 and Theorem 6.3, respectively. The result in (iii) follows from Theorem 6.3(ii) by considering \( 1 - T_\ast \).

To prove the result in (iv), it suffices to show that if \( Y \) has the p.d.f. given by \( \exp \{ \theta y - \zeta_u(\theta) \} \) and if \( u \) is treated as a constant in (1) and (4), \( T_\ast \) in (iii) with a fixed \( u \) is UMP subject to conditions (1) and (4). We now omit \( u \) in the following proof for (iv), which is very similar to the proof of Theorem 6.3.
Proof (continued)

First, \((\alpha, \alpha E_{\theta_0}(Y))\) is an interior point of the set of points \((E_{\theta_0}[T(Y)], E_{\theta_0}[T(Y)Y])\) as \(T\) ranges over all tests of the form \(T(Y)\). By Lemma 6.2 and Proposition 6.1, for testing \(\theta = \theta_0\) versus \(\theta = \theta_1\), the UMP test is equal to 1 when

\[
(k_1 + k_2 y) e^{\theta_0 y} < C(\theta_0, \theta_1) e^{\theta_1 y},
\]

where \(k_i\)'s and \(C(\theta_0, \theta_1)\) are constants.

This inequality is equivalent to

\[
a_1 + a_2 y < e^{by}
\]

for some constants \(a_1, a_2,\) and \(b\).

This region is either one-sided or the outside of an interval.

By Theorem 6.2(ii), a one-sided test has a strictly monotone power function and therefore cannot satisfy (4).

Thus, this test must have the form of \(T_*\) in (iii).

Since \(T_*\) in (iii) does not depend on \(\theta_1\), by Lemma 6.1, it is UMP over all tests satisfying (1) and (4); in particular, the test \(\equiv \alpha\).

Thus, \(T_*\) is UMPU.
Finally, it can be shown that all the $c$- and $\gamma$-functions in (i)-(iv) are Borel functions of $u$ (see Lehmann (1986, p. 149)).