Kolmogorov-Smirnov tests

Let $X_1, \ldots, X_n$ be i.i.d. random variables from a continuous c.d.f. $F$. Consider

$$H_0 : F = F_0 \quad \text{versus} \quad H_1 : F \neq F_0$$

with a fixed $F_0$.

Let $F_n$ be the empirical c.d.f. and

$$D_n(F) = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)|,$$

which is in fact the distance $\rho_\infty(F_n, F)$.

Intuitively, $D_n(F_0)$ should be small if $H_0$ is true.

From the results in §5.1.1, we know that $D_n(F_0) \to a.s. 0$ iff $H_0$ is true.

The statistic $D_n(F_0)$ is called the Kolmogorov-Smirnov statistic. Tests with rejection region $D_n(F_0) > c$ are called Kolmogorov-Smirnov tests.
Kolmogorov-Smirnov tests

In some cases we would like to test “one-sided” hypotheses

\[ H_0 : F = F_0 \quad \text{versus} \quad H_1 : F \geq F_0, \; F \neq F_0, \]

or

\[ H_0 : F = F_0 \quad \text{versus} \quad H_1 : F \leq F_0, \; F \neq F_0. \]

The corresponding Kolmogorov-Smirnov statistic is

\[ D_n^+(F) = \sup_{x \in \mathbb{R}} [F_n(x) - F(x)] \]

or

\[ D_n^-(F) = \sup_{x \in \mathbb{R}} [F(x) - F_n(x)]. \]

The rejection regions of one-sided Kolmogorov-Smirnov tests are, respectively, \( D_n^+(F_0) > c \) and \( D_n^-(F_0) > c \).

Let \( X(1) < \cdots < X(n) \) be the order statistics and define \( X(0) = -\infty \) and \( X(n+1) = \infty \).

Since \( F_n(x) = i/n \) when \( X(i) \leq x < X(i+1), \; i = 0, 1, \ldots, n \),
Kolmogorov-Smirnov tests

In some cases we would like to test “one-sided” hypotheses

\[ H_0 : F = F_0 \quad \text{versus} \quad H_1 : F \geq F_0, \; F \neq F_0, \]

or

\[ H_0 : F = F_0 \quad \text{versus} \quad H_1 : F \leq F_0, \; F \neq F_0. \]

The corresponding Kolmogorov-Smirnov statistic is

\[ D_n^+(F) = \sup_{x \in \mathbb{R}} [F_n(x) - F(x)] \]

or

\[ D_n^-(F) = \sup_{x \in \mathbb{R}} [F(x) - F_n(x)]. \]

The rejection regions of one-sided Kolmogorov-Smirnov tests are, respectively, \( D_n^+(F_0) > c \) and \( D_n^-(F_0) > c \).

Let \( X(1) < \cdots < X(n) \) be the order statistics and define \( X(0) = -\infty \) and \( X(n+1) = \infty \).

Since \( F_n(x) = i/n \) when \( X(i) \leq x < X(i+1), \; i = 0, 1, \ldots, n \),
Kolmogorov-Smirnov tests

\[ D_n^+(F) = \max_{0 \leq i \leq n} \sup_{x(i) \leq x < x(i+1)} \left[ \frac{i}{n} - F(x) \right] \]

\[ = \max_{0 \leq i \leq n} \left[ \frac{i}{n} - \inf_{x(i) \leq x < x(i+1)} F(x) \right] \]

\[ = \max_{0 \leq i \leq n} \left[ \frac{i}{n} - F(X(i)) \right]. \]

When \( F \) is continuous, \( F(X(i)) \) is the \( i \)th order statistic of a sample of size \( n \) from the uniform distribution \( U(0,1) \) irrespective of what \( F \) is. The distribution of \( D_n^+(F) \) does not depend on \( F \), if \( F \) is continuous. The distribution of \( D_n^-(F) \) is the same as that of \( D_n^+(F) \) (exercise). Since

\[ D_n(F) = \max\{D_n^+(F), D_n^-(F)\}, \]

the distribution of \( D_n(F) \) does not depend on \( F \). This means that the distributions of Kolmogorov-Smirnov statistics are known under \( H_0 \) if \( F \) is continuous.
Theorem 6.10 (The distributions of $D_n$, $D_n^+$, and $D_n^-$)

Assume that $F$ is continuous.

(i) For any fixed $n$,

$$P(D_n^+(F) \leq t) = \begin{cases} 
0 & t \leq 0 \\
\frac{n!}{n!} \prod_{i=1}^{n} \int_{\max\left\{0, \frac{n-i+1}{n} - t\right\}}^{u_{n-i+2}} du_1 \cdots du_n & 0 < t < 1 \\
1 & t \geq 1
\end{cases}$$

and

$$P(D_n(F) \leq t) = \begin{cases} 
0 & t \leq \frac{1}{2n} \\
\frac{n!}{n!} \prod_{i=1}^{n} \int_{\max\left\{0, \frac{n-i+1}{n} - t\right\}}^{\min\{u_{n-i+2}, \frac{n-i}{n} + t\}} du_1 \cdots du_n & \frac{1}{2n} < t < 1 \\
1 & t \geq 1
\end{cases}$$

where $u_{n+1} = 1$.

(ii) For $t > 0$,

$$\lim_{n \to \infty} P(\sqrt{n}D_n^+(F) \leq t) = 1 - e^{-2t^2}$$

and

$$\lim_{n \to \infty} P(\sqrt{n}D_n(F) \leq t) = 1 - 2 \sum_{j=1}^{\infty} (-1)^{j-1} e^{-2j^2t^2}.$$
Remarks

- When $n$ is not large, Kolmogorov-Smirnov tests of size $\alpha$ can be obtained using the results in Theorem 6.10(i).
- When $n$ is large, using the results in Theorem 6.10(i) is not convenient. We can obtain Kolmogorov-Smirnov tests of limiting size $\alpha$ using the results in Theorem 6.10(ii).
- It is worthwhile to compare the goodness of fit test introduced in Example 6.23 with the Kolmogorov-Smirnov test.
  - The former requires a partition of the range of observations and may lose information through partitioning, whereas the latter requires that $F$ be continuous and univariate.
  - The latter is of size $\alpha$ (or limiting size $\alpha$), whereas the former is only of asymptotic significance level $\alpha$.
  - The former can be modified to allow estimation of unknown parameters under $H_0$, whereas the latter does not have this flexibility.
Asymptotic tests (tests with asymptotic significance level $\alpha$)

A simple method of constructing asymptotic tests (for almost all problems, parametric or nonparametric) for testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0,$$

where $\theta$ is a vector of parameters, when an asymptotically normally distributed estimator of $\theta$ can be found. However, this simple method may not provide the best or even nearly best solution to the problem, especially when there are different asymptotically normally distributed estimators of $\theta$.

Let $\hat{\theta}_n$ be an estimator of $\theta$ based on a sample of size $n$ from $P$. Suppose that under $H_0$,

$$V_n^{-1/2}(\hat{\theta}_n - \theta) \rightarrow_d N_k(0, I_k),$$

where $V_n$ is the asymptotic covariance matrix of $\hat{\theta}_n$. If $V_n$ is known when $\theta = \theta_0$, then we define a test with rejection region

$$(\hat{\theta}_n - \theta_0)^\tau V_n^{-1}(\hat{\theta}_n - \theta_0) > \chi^2_{k, \alpha},$$

where $\chi^2_{k, \alpha}$ is the $(1 - \alpha)$th quantile of the chi-squared distribution $\chi^2_k$. 
Asymptotic tests (tests with asymptotic significance level $\alpha$)

A simple method of constructing asymptotic tests (for almost all problems, parametric or nonparametric) for testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0,$$

where $\theta$ is a vector of parameters, when an asymptotically normally distributed estimator of $\theta$ can be found. However, this simple method may not provide the best or even nearly best solution to the problem, especially when there are different asymptotically normally distributed estimators of $\theta$.

Let $\hat{\theta}_n$ be an estimator of $\theta$ based on a sample of size $n$ from $P$. Suppose that under $H_0$,

$$V_n^{-1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} N_k(0, I_k),$$

where $V_n$ is the asymptotic covariance matrix of $\hat{\theta}_n$. If $V_n$ is known when $\theta = \theta_0$, then we define a test with rejection region

$$(\hat{\theta}_n - \theta_0)^\tau V_n^{-1}(\hat{\theta}_n - \theta_0) > \chi^2_{k, \alpha},$$

where $\chi^2_{k, \alpha}$ is the $(1 - \alpha)$th quantile of the chi-squared distribution $\chi^2_k$. 
Asymptotic tests (tests with asymptotic significance level $\alpha$)

This test has asymptotic significance level $\alpha$.

If the distribution of $\hat{\theta}_n$ does not depend on the unknown population $P$ under $H_0$, then this test has limiting size $\alpha$.

If $V_n$ depends on the unknown population $P$ even if $H_0$ is true ($\theta = \theta_0$), then we have to replace $V_n$ by an estimator $\hat{V}_n$.

If, under $H_0$, $\hat{V}_n$ is consistent in the sense $\hat{V}_n V_n^{-1} \to_p I$ (Definition 5.4) then the test having the rejection region

$$(\hat{\theta}_n - \theta_0)^\tau \hat{V}_n^{-1} (\hat{\theta}_n - \theta_0) > \chi^2_{k,\alpha}$$

has asymptotic significance level $\alpha$.

Variance estimation methods introduced in §5.5 can be used to construct a consistent estimator $\hat{V}_n$.

The following result shows that, under some additional conditions, the previously defined test is asymptotically correct (§2.5.3), i.e., it is a consistent asymptotic test (Definition 2.13).
Theorem 6.12

Assume that

\[ V_n^{-1/2}(\hat{\theta}_n - \theta) \rightarrow_d N_k(0, I_k), \]

holds for any \( P \).

Assume also that \( \lambda_+ [V_n] \rightarrow 0 \), where \( \lambda_+ [V_n] \) is the largest eigenvalue of \( V_n \).

(i) The test having rejection region

\[ (\hat{\theta}_n - \theta_0) \tau V_n^{-1}(\hat{\theta}_n - \theta_0) > \chi^2_{k,\alpha} \]

with a known \( V_n \) (or with \( V_n \) replaced by a consistent estimator \( \hat{V}_n \)) is consistent.

(ii) If we choose \( \alpha = \alpha_n \rightarrow 0 \) as \( n \rightarrow \infty \) and \( \chi^2_{k,1-\alpha_n} \lambda_+ [V_n] = o(1) \), then the test in (i) is Chernoff-consistent.

Proof

We only prove (i) for the case where \( V_n \) is known.

Let \( Z_n = V_n^{-1/2}(\hat{\theta}_n - \theta) \) and \( l_n = V_n^{-1/2}(\theta - \theta_0) \).

Then \( \|Z_n\| = O_p(1) \) and \( \|l_n\| = \|V_n^{-1/2}(\theta - \theta_0)\| \rightarrow \infty \) when \( \theta \neq \theta_0 \).
Theorem 6.12

Assume that
\[ V_n^{-1/2}(\hat{\theta}_n - \theta) \rightarrow_d N_k(0, I_k), \]
holds for any \( P \).
Assume also that \( \lambda_+[V_n] \rightarrow 0 \), where \( \lambda_+[V_n] \) is the largest eigenvalue of \( V_n \).
(i) The test having rejection region
\[ (\hat{\theta}_n - \theta_0)^\tau V_n^{-1} (\hat{\theta}_n - \theta_0) > \chi^2_{k, \alpha} \]
with a known \( V_n \) (or with \( V_n \) replaced by a consistent estimator \( \hat{V}_n \)) is consistent.
(ii) If we choose \( \alpha = \alpha_n \rightarrow 0 \) as \( n \rightarrow \infty \) and \( \chi^2_{k, 1-\alpha_n} \lambda_+[V_n] = o(1) \), then the test in (i) is Chernoff-consistent.

Proof

We only prove (i) for the case where \( V_n \) is known.
Let \( Z_n = V_n^{-1/2}(\hat{\theta}_n - \theta) \) and \( l_n = V_n^{-1/2}(\theta - \theta_0) \).
Then \( \|Z_n\| = O_p(1) \) and \( \|l_n\| = \|V_n^{-1/2}(\theta - \theta_0)\| \rightarrow \infty \) when \( \theta \neq \theta_0 \).
Proof (continued)

The result follows from the fact that when \( \theta \neq \theta_0 \),

\[
(\hat{\theta}_n - \theta_0)^\tau V_n^{-1}(\hat{\theta}_n - \theta_0) = \left\| Z_n \right\|^2 + \left\| I_n \right\|^2 + 2I_n^\tau Z_n \\
\geq \left\| Z_n \right\|^2 + \left\| I_n \right\|^2 - 2\left\| I_n \right\|\left\| Z_n \right\| \\
= O_p(1) + \left\| I_n \right\|^2[1 - o_p(1)]
\]

and, therefore,

\[
P\left( (\hat{\theta}_n - \theta_0)^\tau V_n^{-1}(\hat{\theta}_n - \theta_0) > \chi^2_{k,\alpha} \right) \rightarrow 1.
\]

Example 6.27

Let \( X_1, \ldots, X_n \) be i.i.d. random variables from a symmetric c.d.f. \( F \) having finite variance and positive \( F' \).
Consider the problem of testing \( H_0 : F \) is symmetric about 0 versus \( H_1 : F \) is not symmetric about 0.
Under \( H_0 \), there are many estimators that are asymptotically normal.
Proof (continued)

The result follows from the fact that when $\theta \neq \theta_0$,

\[
(\hat{\theta}_n - \theta_0)^\tau V_n^{-1}(\hat{\theta}_n - \theta_0) = \|Z_n\|^2 + \|l_n\|^2 + 2l_n^\tau Z_n \\
\geq \|Z_n\|^2 + \|l_n\|^2 - 2\|l_n\|\|Z_n\| \\
= O_p(1) + \|l_n\|^2[1 - o_p(1)]
\]

and, therefore,

\[
P \left( (\hat{\theta}_n - \theta_0)^\tau V_n^{-1}(\hat{\theta}_n - \theta_0) > \chi^2_k, \alpha \right) \to 1.
\]

Example 6.27

Let $X_1, \ldots, X_n$ be i.i.d. random variables from a symmetric c.d.f. $F$ having finite variance and positive $F'$.

Consider the problem of testing $H_0 : F$ is symmetric about 0 versus $H_1 : F$ is not symmetric about 0.

Under $H_0$, there are many estimators that are asymptotically normal.
We consider the following three estimators:

1. \( \hat{\theta}_n = \bar{X} \) and \( \theta = E(X_1) \);
2. \( \hat{\theta}_n = \hat{\theta}_{0.5} \) (the sample median) and \( \theta = F^{-1}(\frac{1}{2}) \) (the median of \( F \));
3. \( \hat{\theta}_n = \bar{X}_a \) (the \( a \)-trimmed sample mean) and \( \theta = \int xJ(F(x))dF(x) \) with \( J(t) = (1 - 2a)^{-1}I_{(a, 1-a)}(t) \), \( a \in (0, \frac{1}{2}) \).

Although the \( \theta \)'s in (1)-(3) are different in general, in all cases \( \theta = 0 \) is equivalent to that \( H_0 \) holds.

For \( \bar{X} \), it follows from the CLT that

\[
V_n^{-1/2}(\bar{X} - \theta) \xrightarrow{d} N(0, 1)
\]

with \( V_n = \sigma^2/n \) for any \( F \), where \( \sigma^2 = \text{Var}(X_1) \).

From the SLLN, \( S^2/n \) is a consistent estimator of \( V_n \) for any \( F \).

Thus, Theorem 6.12 applies with \( \hat{\theta}_n = \bar{X} \) and \( V_n \) replaced by \( S^2/n \).

This test is asymptotically equivalent to the one-sample t-test derived in §6.2.3.
Example 6.27 (continued)

We consider the following three estimators:

1. \( \hat{\theta}_n = \bar{X} \) and \( \theta = E(X_1) \);
2. \( \hat{\theta}_n = \hat{\theta}_{0.5} \) (the sample median) and \( \theta = F^{-1}(1/2) \) (the median of \( F \));
3. \( \hat{\theta}_n = \bar{X}_a \) (the \( a \)-trimmed sample mean) and
   \[
   \theta = \int x J(F(x)) dF(x)
   \]
   with \( J(t) = (1 - 2a)^{-1} l_{(a, 1-a)}(t), \ a \in (0, 1/2) \).

Although the \( \theta \)'s in (1)-(3) are different in general, in all cases \( \theta = 0 \) is equivalent to that \( H_0 \) holds.

For \( \bar{X} \), it follows from the CLT that

\[
V_n^{-1/2} (\bar{X} - \theta) \to_d N(0, 1)
\]

with \( V_n = \sigma^2/n \) for any \( F \), where \( \sigma^2 = \text{Var}(X_1) \).

From the SLLN, \( S^2/n \) is a consistent estimator of \( V_n \) for any \( F \).

Thus, Theorem 6.12 applies with \( \hat{\theta}_n = \bar{X} \) and \( V_n \) replaced by \( S^2/n \).

This test is asymptotically equivalent to the one-sample t-test derived in §6.2.3.
Example 6.27 (continued)

From Theorem 5.10, \( \hat{\theta}_{0.5} \) satisfies
\[
V_n^{-1/2}(\hat{\theta} - \theta) \to_d N(0, 1)
\]
with \( V_n = 4^{-1}[F'(\theta)]^{-2}n^{-1} \) for any \( F \).

A consistent estimator of \( V_n \) can be obtained using the bootstrap method considered in §5.5.3.

Another consistent estimator of \( V_n \) can be obtained using Woodruff’s interval introduced in §7.4 (see Exercise 86 in §7.6).

Thus, Theorem 6.12 applies with \( \hat{\theta}_n = \hat{\theta}_{0.5} \) and \( V_n \) replaced by a consistent estimator.

It follows from the discussion in §5.3.2 that \( \bar{X}_a \) satisfies
\[
V_n^{-1/2}(\bar{X}_a - \theta) \to_d N(0, 1)
\]
A consistent estimator of \( V_n \) can be obtained using the formula for \( \sigma_a^2 \).

Thus, Theorem 6.12 applies with \( \hat{\theta}_n = \bar{X}_a \) and \( V_n \) replaced by a consistent estimator is asymptotically correct.
Example 6.27 (continued)

From Theorem 5.10, $\hat{\theta}_{0.5}$ satisfies

$$V_n^{-1/2} (\hat{\theta} - \theta) \xrightarrow{d} N(0, 1)$$

with $V_n = 4^{-1}[F'(\theta)]^{-2} n^{-1}$ for any $F$.

A consistent estimator of $V_n$ can be obtained using the bootstrap method considered in §5.5.3.

Another consistent estimator of $V_n$ can be obtained using Woodruff’s interval introduced in §7.4 (see Exercise 86 in §7.6).

Thus, Theorem 6.12 applies with $\hat{\theta}_n = \hat{\theta}_{0.5}$ and $V_n$ replaced by a consistent estimator.

It follows from the discussion in §5.3.2 that $\bar{X}_a$ satisfies

$$V_n^{-1/2} (\bar{X}_a - \theta) \xrightarrow{d} N(0, 1)$$

A consistent estimator of $V_n$ can be obtained using the formula for $\sigma_a^2$.

Thus, Theorem 6.12 applies with $\hat{\theta}_n = \bar{X}_a$ and $V_n$ replaced by a consistent estimator is asymptotically correct.
Example 6.27 (continued)

It is not clear which one of the tests discussed here is to be preferred in general.

The results for \( \hat{\theta}_n \) in (1)-(3) still hold for testing \( H_0 : \theta = 0 \) versus \( H_1 : \theta \neq 0 \) without the assumption that \( F \) is symmetric.

An example of asymptotic tests for one-sided hypotheses is given in Exercise 123.
Example 6.27 (continued)

It is not clear which one of the tests discussed here is to be preferred in general. The results for $\hat{\theta}_n$ in (1)-(3) still hold for testing $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$ without the assumption that $F$ is symmetric.

An example of asymptotic tests for one-sided hypotheses is given in Exercise 123.