Confidence sets and hypothesis tests

Another popular method of constructing confidence sets is to use a close relationship between confidence sets and hypothesis tests. For any test $T$, the set $\{x : T(x) \neq 1\}$ is called the acceptance region. This terminology is not precise when $T$ is a randomized test.

**Theorem 7.2**

For each $\theta_0 \in \Theta$, let $T_{\theta_0}$ be a test for $H_0 : \theta = \theta_0$ (versus some $H_1$) with significance level $\alpha$ and acceptance region $A(\theta_0)$. For each $x$ in the range of $X$, define

$$C(x) = \{\theta : x \in A(\theta)\}.$$ 

Then $C(X)$ is a level $1 - \alpha$ confidence set for $\theta$. If $T_{\theta_0}$ is nonrandomized and has size $\alpha$ for every $\theta_0$, then $C(X)$ has confidence coefficient $1 - \alpha$. 
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Proof

We prove the first assertion only. The proof for the second assertion is similar. Under the given condition,

\[
\sup_{\theta=\theta_0} P(X \not\in A(\theta_0)) = \sup_{\theta=\theta_0} P(T_{\theta_0} = 1) \leq \alpha,
\]

which is the same as

\[
1 - \alpha \leq \inf_{\theta=\theta_0} P(X \in A(\theta_0)) = \inf_{\theta=\theta_0} P(\theta_0 \in C(X)).
\]

Since this holds for all \( \theta_0 \), the result follows from

\[
\inf_{P \in \mathcal{P}} P(\theta \in C(X)) = \inf_{\theta_0 \in \Theta} \inf_{\theta=\theta_0} P(\theta_0 \in C(X)) \geq 1 - \alpha.
\]

The converse of Theorem 7.2 is partially true.
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The converse of Theorem 7.2 is partially true.
Proposition 7.2

Let $C(X)$ be a confidence set for $\theta$ with confidence level (or confidence coefficient) $1 - \alpha$.

For any $\theta_0 \in \Theta$, define a region $A(\theta_0) = \{x : \theta_0 \in C(x)\}$.

Then the test $T(X) = 1 - I_{A(\theta_0)}(X)$ has significance level $\alpha$ for testing $H_0 : \theta = \theta_0$ versus some $H_1$.

Discussions

In general, $C(X)$ in Theorem 7.2 can be determined numerically, if it does not have an explicit form.

Suppose $A(\theta) = \{Y : a(\theta) \leq Y \leq b(\theta)\}$ for a real-valued $\theta$ and statistic $Y(X)$ and some nondecreasing functions $a(\theta)$ and $b(\theta)$.

When we observe $Y = y$, $C(X)$ is an interval with limits $\underline{\theta}$ and $\overline{\theta}$, which are the $\theta$-values at which the horizontal line $Y = y$ intersects the curves $Y = b(\theta)$ and $Y = a(\theta)$ (Figure 7.1), respectively.

If $y = b(\theta)$ (or $y = a(\theta)$) has no solution or more than one solution, $\underline{\theta} = \inf\{\theta : y \leq b(\theta)\}$ (or $\overline{\theta} = \sup\{\theta : a(\theta) \leq y\}$).

$C(X)$ does not include $\underline{\theta}$ (or $\overline{\theta}$) if and only if at $\underline{\theta}$ (or $\overline{\theta}$), $b(\theta)$ (or $a(\theta)$) is only left-continuous (or right-continuous).
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For any $\theta_0 \in \Theta$, define a region $A(\theta_0) = \{ x : \theta_0 \in C(x) \}$.

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Discussions

In general, $C(X)$ in Theorem 7.2 can be determined numerically, if it does not have an explicit form.

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$C(X)$ does not include $\underline{\theta}$ (or $\overline{\theta}$) if and only if at $\underline{\theta}$ (or $\overline{\theta}$), $b(\theta)$ (or $a(\theta)$) is only left-continuous (or right-continuous).
Figure 7.1. A confidence interval obtained by inverting $A(\theta) = [a(\theta), b(\theta)]$
Example 7.7

Suppose that $X$ has the following p.d.f. in a one-parameter exponential family:

$$f_\theta(x) = \exp\{\eta(\theta)Y(x) - \xi(\theta)\}h(x),$$

where $\theta$ is real-valued and $\eta(\theta)$ is nondecreasing in $\theta$.

First, we apply Theorem 7.2 with $H_0 : \theta = \theta_0$ and $H_1 : \theta > \theta_0$.

By Theorem 6.2, the acceptance region of the UMP test of size $\alpha$ is

$$A(\theta_0) = \{x : Y(x) \leq c(\theta_0)\},$$

where $c(\theta_0) = c$ in Theorem 6.2.

It can be shown that $c(\theta)$ is nondecreasing in $\theta$.

Inverting $A(\theta)$ according to Figure 7.1 with $b(\theta) = c(\theta)$ and $a(\theta)$ ignored, we obtain

$$C(X) = [\underline{\theta}(X), \infty) \text{ or } (\overline{\theta}(X), \infty),$$

a one-sided confidence interval for $\theta$ with confidence level $1 - \alpha$.

$\underline{\theta}(X)$ is called a lower confidence bound for $\theta$ in §2.4.3.

When the c.d.f. of $Y(X)$ is continuous, $C(X)$ has confidence coefficient $1 - \alpha$. 
Example 7.7 (continued)

If \( H_0 : \theta = \theta_0 \) and \( H_1 : \theta < \theta_0 \) are considered, then \( C(X) = \{ \theta : Y(X) \geq c(\theta) \} \) and is of the form

\[
(-\infty, \bar{\theta}(X)] \quad \text{or} \quad (-\infty, \bar{\theta}(X)).
\]

\( \bar{\theta}(X) \) is called an upper confidence bound for \( \theta \).

Consider next \( H_0 : \theta = \theta_0 \) and \( H_1 : \theta \neq \theta_0 \).

By Theorem 6.4, the acceptance region of the UMPU test of size \( \alpha \) is given by \( A(\theta_0) = \{ x : c_1(\theta_0) \leq Y(x) \leq c_2(\theta_0) \} \), where \( c_i(\theta) \) are nondecreasing (exercise).

A confidence interval can be obtained by inverting \( A(\theta) \) according to Figure 7.1 with \( a(\theta) = c_1(\theta) \) and \( b(\theta) = c_2(\theta) \).

Let us consider a specific example in which \( X_1, \ldots, X_n \) are i.i.d. binary random variables with \( p = P(X_i = 1) \).

Note that \( Y(X) = \sum_{i=1}^n X_i \).

Suppose that we need a lower confidence bound for \( p \) so that we consider \( H_0 : p = p_0 \) and \( H_1 : p > p_0 \).
Example 7.7 (continued)

If $H_0 : \theta = \theta_0$ and $H_1 : \theta < \theta_0$ are considered, then $C(X) = \{ \theta : Y(X) \geq c(\theta) \}$ and is of the form

$$(-\infty, \overline{\theta}(X)] \quad \text{or} \quad (-\infty, \overline{\theta}(X)).$$

$\overline{\theta}(X)$ is called an upper confidence bound for $\theta$.

Consider next $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$.

By Theorem 6.4, the acceptance region of the UMPU test of size $\alpha$ is given by $A(\theta_0) = \{ x : c_1(\theta_0) \leq Y(x) \leq c_2(\theta_0) \}$, where $c_i(\theta)$ are nondecreasing (exercise).

A confidence interval can be obtained by inverting $A(\theta)$ according to Figure 7.1 with $a(\theta) = c_1(\theta)$ and $b(\theta) = c_2(\theta)$.

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Note that $Y(X) = \sum_{i=1}^{n} X_i$.

Suppose that we need a lower confidence bound for $p$ so that we consider $H_0 : p = p_0$ and $H_1 : p > p_0$. 
If $H_0 : \theta = \theta_0$ and $H_1 : \theta < \theta_0$ are considered, then $C(X) = \{ \theta : Y(X) \geq c(\theta) \}$ and is of the form

$(-\infty, \bar{\theta}(X)]$ or $(-\infty, \bar{\theta}(X))$.

$\bar{\theta}(X)$ is called an upper confidence bound for $\theta$.

Consider next $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$.

By Theorem 6.4, the acceptance region of the UMPU test of size $\alpha$ is given by $A(\theta_0) = \{ x : c_1(\theta_0) \leq Y(x) \leq c_2(\theta_0) \}$, where $c_i(\theta)$ are nondecreasing (exercise).

A confidence interval can be obtained by inverting $A(\theta)$ according to Figure 7.1 with $a(\theta) = c_1(\theta)$ and $b(\theta) = c_2(\theta)$.

Let us consider a specific example in which $X_1, \ldots, X_n$ are i.i.d. binary random variables with $p = P(X_i = 1)$.

Note that $Y(X) = \sum_{i=1}^n X_i$.

Suppose that we need a lower confidence bound for $p$ so that we consider $H_0 : p = p_0$ and $H_1 : p > p_0$. 
Example 7.7 (continued)

From Example 6.2, the acceptance region of a UMP test of size \( \alpha \in (0, 1) \) is \( A(p_0) = \{ y : y \leq m(p_0) \} \), where \( m(p_0) \) is an integer between 0 and \( n \) such that

\[
\sum_{j=m(p_0)+1}^{n} \binom{n}{j} p_0^j (1 - p_0)^{n-j} \leq \alpha < \sum_{j=m(p_0)}^{n} \binom{n}{j} p_0^j (1 - p_0)^{n-j}.
\]

Thus, \( m(p) \) is an integer-valued, right-continuous, nondecreasing step-function of \( p \).

Define

\[
p = \inf \{ p : m(p) \geq y \} = \inf \left\{ p : \sum_{j=y}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \geq \alpha \right\}.
\]

We want to show that a level \( 1 - \alpha \) confidence interval for \( p \) is \( (\underline{p}, 1] \).

Inverting \( A(p) \) we obtain that

\[
C(y) = \{ p : y \leq m(p) \}
\]
Example 7.7 (continued)

We need to show that

\[ \{ p : y \leq m(p) \} = \{ p : \underline{p} < p \} \]

Suppose that \( \underline{p} < p \). If \( m(p) < y \), then, by the definition of \( \underline{p} \), we must have \( p \leq \underline{p} \), a contradiction. Hence, we must have \( y \leq m(p) \). This shows

\[ \{ p : \underline{p} < p \} \subset \{ p : y \leq m(p) \} \]

Suppose that \( y \leq m(p) \). By the definition of \( \underline{p} \), \( p \leq \underline{p} \). But we cannot have \( \underline{p} = p \), because \( m(p) \) is right-continuous and it jumps up at \( \underline{p} \). Thus, \( \underline{p} < p \) and, hence,

\[ \{ p : y \leq m(p) \} \subset \{ p : \underline{p} < p \} \]

One can compare this confidence interval with the one obtained by applying Theorem 7.1 (exercise).
See also Example 7.16.
Example 7.8

Suppose that $X$ has the following p.d.f. in a multiparameter exponential family:

$$f_{\theta, \varphi}(x) = \exp \{ \theta Y(x) + \varphi^\tau U(x) - \zeta(\theta, \varphi) \}$$

By Theorem 6.4, the acceptance region of a UMPU test of size $\alpha$ for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ or $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is

$$A(\theta_0) = \{ (y, u) : y \leq c_2(u, \theta_0) \}$$

or

$$A(\theta_0) = \{ (y, u) : c_1(u, \theta_0) \leq y \leq c_2(u, \theta_0) \},$$

where $c_i(u, \theta), i = 1, 2,$ are nondecreasing functions of $\theta$. Confidence intervals for $\theta$ can then be obtained by inverting $A(\theta)$ according to Figure 7.1 with $b(\theta) = c_2(u, \theta)$ and $a(\theta) = c_1(u, \theta)$ or $a(\theta) \equiv -\infty$, for any observed $u$. 
Example 7.8 (continued)

Consider more specifically the case where $X_1$ and $X_2$ are independently distributed as the Poisson distributions $P(\lambda_1)$ and $P(\lambda_2)$, respectively, and we need a lower confidence bound for the ratio $\rho = \lambda_2/\lambda_1$.

From Example 6.11, a UMPU test of size $\alpha$ for testing $H_0 : \rho = \rho_0$ versus $H_1 : \rho > \rho_0$ has the acceptance region

$$A(\rho_0) = \{(y, u) : y \leq c(u, \rho_0)\},$$

where $c(u, \rho_0)$ is determined by the conditional distribution of $Y = X_2$ given $U = X_1 + X_2 = u$.

Since the conditional distribution of $Y$ given $U = u$ is the binomial distribution $Bi(\rho/(1 + \rho), u)$, we can use the result in Example 7.7, i.e., $c(u, \rho)$ is the same as $m(\rho)$ in Example 7.7 with $n = u$ and $p = \rho/(1 + \rho)$.
Example 7.8 (continued)

Then a level $1 - \alpha$ lower confidence bound for $p$ is $p$ given by

$$p = \inf \{ p : m(p) \geq y \} = \inf \left\{ p : \sum_{j=y}^{u} \binom{u}{j} p^j (1-p)^{u-j} \geq \alpha \right\}$$

Since $\rho = p/(1-p)$ is a strictly increasing function of $p$, a level $1 - \alpha$ lower confidence bound for $\rho$ is $p/(1-p)$.

Example 7.9

Consider the normal linear model $X = N_n(Z\beta, \sigma^2 I_n)$ and the problem of constructing a confidence set for $\theta = L\beta$, where $L$ is an $s \times p$ matrix of rank $s$ and all rows of $L$ are in $\mathcal{R}(Z)$.

The LR test of size $\alpha$ for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has the acceptance region

$$A(\theta_0) = \{ X : W(X, \theta_0) \leq c_\alpha \},$$
Example 7.8 (continued)

Then a level $1 - \alpha$ lower confidence bound for $p$ is $\underline{p}$ given by

$$\underline{p} = \inf \{ p : m(p) \geq y \} = \inf \left\{ p : \sum_{j=y}^{u} \binom{u}{j} p^j (1 - p)^{u-j} \geq \alpha \right\}$$

Since $\rho = p/(1 - p)$ is a strictly increasing function of $p$, a level $1 - \alpha$ lower confidence bound for $\rho$ is $\underline{p}/(1 - \underline{p})$.

Example 7.9

Consider the normal linear model $X = N_n(Z\beta, \sigma^2 I_n)$ and the problem of constructing a confidence set for $\theta = L\beta$, where $L$ is an $s \times p$ matrix of rank $s$ and all rows of $L$ are in $R(Z)$.

The LR test of size $\alpha$ for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has the acceptance region

$$A(\theta_0) = \{ X : W(X, \theta_0) \leq c_\alpha \},$$
Example 7.9 (continued)

where $c_\alpha$ is the $(1 - \alpha)$th quantile of the F-distribution $F_{s,n-r},$

$$W(X, \theta) = \frac{[\|X - Z\hat{\beta}(\theta)\|^2 - \|X - Z\hat{\beta}\|^2]/s}{\|X - Z\hat{\beta}\|^2/(n-r)},$$

$r$ is the rank of $Z$, $r \geq s$, $\hat{\beta}$ is the LSE of $\beta$ and, for each fixed $\theta$, $\hat{\beta}(\theta)$ is a solution of

$$\|X - Z\hat{\beta}(\theta)\|^2 = \min_{\beta: L\beta = \theta} \|X - Z\beta\|^2.$$

Inverting $A(\theta)$, we obtain the following confidence set for $\theta$ with confidence coefficient $1 - \alpha$: $C(X) = \{\theta : W(X, \theta) \leq c_\alpha\}$, which forms a closed ellipsoid in $\mathbb{R}^s$. 