We consider another example of asymptotic confidence sets based on likelihood discussed in the last lecture.

**Example 7.24**

Let $X_1, ..., X_n$ be i.i.d. from $N(\mu, \varphi)$ with unknown $\theta = (\mu, \varphi)$. Consider the problem of constructing a $1 - \alpha$ asymptotically correct confidence set for $\theta$.

The log-likelihood function is

$$
\log \ell(\theta) = -\frac{1}{2\varphi} \sum_{i=1}^{n} (X_i - \mu)^2 - \frac{n}{2} \log \varphi - \frac{n}{2} \log (2\pi).
$$

Since $(\bar{X}, \hat{\varphi})$ is the MLE of $\theta$, where $\hat{\varphi} = (n - 1)S^2/n$, the confidence set based on LR tests is

$$
C_1(X) = \left\{ \theta : \frac{1}{\varphi} \sum_{i=1}^{n} (X_i - \mu)^2 + n \log \varphi \leq \chi^2_{2, \alpha} + n + n \log \hat{\varphi} \right\}.
$$
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Example 7.24 (continued)

Note that

\[ s_n(\theta) = \left( \frac{n(\bar{X} - \mu)}{\phi}, \frac{1}{2\phi^2} \sum_{i=1}^{n} (X_i - \mu)^2 - \frac{n}{2\phi} \right) \quad I_n(\theta) = \begin{pmatrix} \frac{n}{\phi} & 0 \\ 0 & \frac{n}{2\phi^2} \end{pmatrix}. \]

Hence, the confidence set based on Wald’s tests is

\[ C_2(X) = \left\{ \theta : \frac{(\bar{X} - \mu)^2}{\hat{\phi}} + \frac{(\hat{\phi} - \phi)^2}{2\hat{\phi}^2} \leq \frac{\chi^2_{2,\alpha}}{n} \right\}, \]

which is an ellipsoid in \( \mathbb{R}^2 \), and the confidence set based on Rao’s score tests is

\[ C_3(X) = \left\{ \theta : \frac{(\bar{X} - \mu)^2}{\phi} + \frac{1}{2} \left[ \frac{1}{n\phi} \sum_{i=1}^{n} (X_i - \mu)^2 - 1 \right]^2 \leq \frac{\chi^2_{2,\alpha}}{n} \right\}. \]

In general, \( C_j(X), j = 1,2,3, \) are different.
An example of these three confidence sets is given in Figure 7.2, where \( n=100, \mu=0, \) and \( \phi=1. \)
Figure 7.2. Confidence sets obtained by inverting LR, Wald’s, and Rao’s score tests in Example 7.24
Example 7.24 (continued)

Consider now the construction of a confidence set for \( \mu \).

The confidence set based on Wald’s tests is defined by \( C_2(X) \) with \( \phi \) replaced by \( \hat{\phi} \) and \( \chi^2_{2,\alpha} \) replaced by \( \chi^2_{1,\alpha} = z^2_{\alpha/2} \), which results in the confidence interval

\[
\{ \mu : n(\bar{X} - \mu)^2 \leq z^2_{\alpha/2}\hat{\phi} \} = [\bar{X} - z_{\alpha/2}\sqrt{\hat{\phi}/n}, \bar{X} + z_{\alpha/2}\sqrt{\hat{\phi}/n}]
\]

The confidence set based on the LR tests is defined by \( C_1(X) \) with \( \chi^2_{2,\alpha} \) and \( \phi \) replaced by \( \chi^2_{1,\alpha} = z^2_{\alpha/2} \) and \( n^{-1}\sum_{i=1}^{n}(X_i - \mu)^2 = \hat{\phi} + (\bar{X} - \mu)^2 \), respectively, which leads to the confidence interval

\[
\{ \mu : n + n\log(\hat{\phi} + (\bar{X} - \mu)^2) \leq z^2_{\alpha/2} + n + n\log \hat{\phi} \}
\]

\[
= \{ \mu : \hat{\phi} + (\bar{X} - \mu)^2 \leq \exp(\log \hat{\phi} + z^2_{\alpha/2}/n) \}
\]

\[
= \{ \mu : (\bar{X} - \mu)^2 \leq \hat{\phi}[\exp(z^2_{\alpha/2}/n) - 1] \}
\]

\[
= [\bar{X} - \sqrt{\hat{\phi}}\sqrt{\exp(z^2_{\alpha/2}/n) - 1}, \bar{X} + \sqrt{\hat{\phi}}\sqrt{\exp(z^2_{\alpha/2}/n) - 1}]
\]
The confidence set based on Rao’s score tests is defined by $C_3(X)$ with $\chi^2_{2,\alpha}$ and $\varphi$ replaced by $z^2_{\alpha/2}$ and $n^{-1} \sum_{i=1}^{n} (X_i - \mu)^2 = \hat{\varphi} + (\bar{X} - \mu)^2$, respectively, which results in the confidence interval

$$\{ \mu : n(\bar{X} - \mu)^2 \leq z^2_{\alpha/2}[\hat{\varphi} + (\bar{X} - \mu)^2] \}$$

$$= [\bar{X} - z_{\alpha/2} \sqrt{\hat{\varphi}/n} \sqrt{1 - z^2_{\alpha/2}/n}, \bar{X} + z_{\alpha/2} \sqrt{\hat{\varphi}/n} \sqrt{1 - z^2_{\alpha/2}/n}]$$

Confidence intervals for quantiles

Let $X_1, \ldots, X_n$ be i.i.d. from a continuous c.d.f. $F$ on $\mathbb{R}$ and let $\theta = F^{-1}(p)$ be the $p$th quantile of $F$, $0 < p < 1$. The general methods we previously discussed can be applied to obtain a confidence set for $\theta$, but we introduce here a method that works particularly for quantile problems.

In fact, for any given $\alpha$, it is possible to derive a confidence interval (or bound) for $\theta$ with confidence coefficient $1 - \alpha$ (Exercise 84), but the computation of such a confidence interval may be cumbersome. We focus on asymptotic confidence intervals for $\theta$.

Our result is based on the following result due to Bahadur (1966).
Example 7.24 (continued)

The confidence set based on Rao’s score tests is defined by $C_3(X)$ with $\chi^2_{2,\alpha}$ and $\phi$ replaced by $z^2_{\alpha/2}$ and $n^{-1} \sum_{i=1}^n (X_i - \mu)^2 = \hat{\phi} + (\bar{X} - \mu)^2$, respectively, which results in the confidence interval

$$\{\mu : n(\bar{X} - \mu)^2 \leq z^2_{\alpha/2}[\hat{\phi} + (\bar{X} - \mu)^2]\}$$

$$= [\bar{X} - z_{\alpha/2}\sqrt{\hat{\phi}/n}\sqrt{1 - z^2_{\alpha/2}/n}, \bar{X} + z_{\alpha/2}\sqrt{\hat{\phi}/n}\sqrt{1 - z^2_{\alpha/2}/n}]$$

Confidence intervals for quantiles

Let $X_1, ..., X_n$ be i.i.d. from a continuous c.d.f. $F$ on $\mathbb{R}$ and let $\theta = F^{-1}(p)$ be the $p$th quantile of $F$, $0 < p < 1$.

The general methods we previously discussed can be applied to obtain a confidence set for $\theta$, but we introduce here a method that works particularly for quantile problems.

In fact, for any given $\alpha$, it is possible to derive a confidence interval (or bound) for $\theta$ with confidence coefficient $1 - \alpha$ (Exercise 84), but the computation of such a confidence interval may be cumbersome.

We focus on asymptotic confidence intervals for $\theta$.

Our result is based on the following result due to Bahadur (1966).
Theorem 7.8

Let \( X_1, \ldots, X_n \) be i.i.d. from a continuous c.d.f. \( F \) on \( \mathbb{R} \) that is twice differentiable at \( \theta = F^{-1}(p) \), \( 0 < p < 1 \), with \( F'(\theta) > 0 \).

Let \( F_n \) be the empirical c.d.f.

Let \( \{k_n\} \) be a sequence of integers satisfying \( 1 \leq k_n \leq n \) and \( k_n/n = p + o((\log n)^\delta / \sqrt{n}) \) for some \( \delta > 0 \).

Then

\[
X_{(k_n)} = \theta + \frac{(k_n/n) - F_n(\theta)}{F'(\theta)} + O \left( \frac{(\log n)^{(1+\delta)/2}}{n^{3/4}} \right) \quad \text{a.s.}
\]

Proof

Omitted.

The result in Theorem 7.8 is a refinement of the Bahadur representation in Theorem 5.11.

The following corollary of Theorem 7.8 is useful in statistics.
Let \( \hat{\theta}_n = F_n^{-1}(p) \) be the sample \( p \)th quantile.
Theorem 7.8

Let $X_1, \ldots, X_n$ be i.i.d. from a continuous c.d.f. $F$ on $\mathbb{R}$ that is twice differentiable at $\theta = F^{-1}(p)$, $0 < p < 1$, with $F'(\theta) > 0$.
Let $F_n$ be the empirical c.d.f.
Let $\{k_n\}$ be a sequence of integers satisfying $1 \leq k_n \leq n$ and $k_n/n = p + o\left((\log n)^\delta/\sqrt{n}\right)$ for some $\delta > 0$.
Then

$$X_{(k_n)} = \theta + \frac{(k_n/n) - F_n(\theta)}{F'(\theta)} + O\left(\frac{\log n^{(1+\delta)/2}}{n^{3/4}}\right) \quad \text{a.s.}$$

Proof

Omitted.

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Let $X_1, \ldots, X_n$ be i.i.d. from a continuous c.d.f. $F$ on $\mathbb{R}$ that is twice differentiable at $\theta = F^{-1}(p)$, $0 < p < 1$, with $F'(\theta) > 0$. Let $F_n$ be the empirical c.d.f. Let $\{k_n\}$ be a sequence of integers satisfying $1 \leq k_n \leq n$ and $k_n/n = p + o((\log n)^\delta/\sqrt{n})$ for some $\delta > 0$. Then

$$X_{(k_n)} = \theta + \frac{(k_n/n) - F_n(\theta)}{F'(\theta)} + O\left(\frac{(\log n)^{(1+\delta)/2}}{n^{3/4}}\right) \text{ a.s.}$$

Proof

Omitted.

The result in Theorem 7.8 is a refinement of the Bahadur representation in Theorem 5.11.

The following corollary of Theorem 7.8 is useful in statistics. Let $\hat{\theta}_n = F_n^{-1}(p)$ be the sample $p$th quantile.
Corollary 7.1

Assume the conditions in Theorem 7.8 and \( k_n/n = p + cn^{-1/2} + o(n^{-1/2}) \) with a constant \( c \).

Then

\[
\sqrt{n}(X(\mathcal{I}_n) - \hat{\theta}_n) \to_{a.s.} c/F'(\theta).
\]

Proof

Left as an exercise.

Using Corollary 7.1, we can obtain a confidence interval for \( \theta \) with limiting confidence coefficient \( 1 - \alpha \) (Definition 2.14) for any given \( \alpha \in (0, 1/2) \).

This is stated and proved in the next result.
Corollary 7.1
Assume the conditions in Theorem 7.8 and $k_n/n = p + cn^{-1/2} + o(n^{-1/2})$ with a constant $c$. Then

$$\sqrt{n}(X_{(k_n)} - \hat{\theta}_n) \to_{a.s.} c/F'(\theta).$$

Proof
Left as an exercise.

Using Corollary 7.1, we can obtain a confidence interval for $\theta$ with limiting confidence coefficient $1 - \alpha$ (Definition 2.14) for any given $\alpha \in (0, \frac{1}{2})$. This is stated and proved in the next result.
Corollary 7.1

Assume the conditions in Theorem 7.8 and 

\[ \frac{k_n}{n} = p + cn^{-1/2} + o(n^{-1/2}) \]

with a constant \( c \).

Then

\[ \sqrt{n}(X(k_n) - \hat{\theta}_n) \xrightarrow{a.s.} c/F'(\theta). \]

Proof

Left as an exercise.

Using Corollary 7.1, we can obtain a confidence interval for \( \theta \) with limiting confidence coefficient \( 1 - \alpha \) (Definition 2.14) for any given \( \alpha \in (0, \frac{1}{2}) \).

This is stated and proved in the next result.
Corollary 7.2

Assume the conditions in Theorem 7.8.
Let \( \{k_{1n}\} \) and \( \{k_{2n}\} \) be two sequences of integers satisfying
\[
1 \leq k_{1n} < k_{2n} \leq n,
\]

\[
k_{1n}/n = p - z_{1-\alpha/2} \sqrt{p(1-p)/n} + o(n^{-1/2}),
\]

and
\[
k_{2n}/n = p + z_{1-\alpha/2} \sqrt{p(1-p)/n} + o(n^{-1/2}),
\]

where \( z_a = \Phi^{-1}(a) \). Then the confidence interval \( C(X) = [X(k_{1n}), X(k_{2n})] \) has the property that \( P(\theta \in C(X)) \) does not depend on \( P \) and

\[
\lim_{n \to \infty} \inf_{P \in \mathcal{P}} P(\theta \in C(X)) = \lim_{n \to \infty} P(\theta \in C(X)) = 1 - \alpha.
\]

Furthermore,

\[
\text{the length of } C(X) = \frac{2z_{1-\alpha/2} \sqrt{p(1-p)}}{F'(\theta) \sqrt{n}} + o \left(\frac{1}{\sqrt{n}}\right) \text{ a.s.}
\]
Proof

Note that $P(\theta \in C(X)) = P(X_{(k_1n)} \leq \theta \leq X_{(k_2n)}) = P(U_{(k_1n)} \leq \rho \leq U_{(k_2n)})$, where $U_{(k)}$ is the $k$th order statistic based on a sample $U_1, ..., U_n$ i.i.d. from the uniform distribution $U(0, 1)$ (Exercise 84). Hence, $P(\theta \in C(X))$ does not depend on $P$ and
\[
\lim_{n \to \infty} P(\theta \in C(X)) = \lim_{n \to \infty} \inf_{\mathcal{P}} P(\theta \in C(X)).
\]
By Corollary 7.1, Theorem 5.10, and Slutsky’s theorem,
\[
P(X_{(k_1n)} > \theta) = P\left(\sqrt{n}(\hat{\theta}_n - \theta) \sqrt{p(1-p)}/F'(\theta) + o_p(n^{-1/2}) > \theta\right)
= P\left(\sqrt{n}(\hat{\theta}_n - \theta) \sqrt{p(1-p)/F'(\theta)} + o_p(1) > z_{1-\alpha/2}\right)
\to 1 - \Phi(z_{1-\alpha/2})
= \alpha/2.
\]

The first result follows, since similarly $P(X_{(k_2n)} < \theta) \to \alpha/2$. The result for the length of $C(X)$ follows directly from Corollary 7.1.
The confidence interval $[X_{(k_1 n)}, X_{(k_2 n)}]$ given in Corollary 7.2 is called Woodruff’s (1952) interval. It has limiting confidence coefficient $1 - \alpha$, a property that is stronger than the $1 - \alpha$ asymptotic correctness. The length of Woodruff’s interval is $X_{(k_2 n)} - X_{(k_1 n)}$. By the result in Corollary 7.2,

$$X_{(k_2 n)} - X_{(k_1 n)} = \frac{2z_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{nF'(\theta)}} + o\left(\frac{1}{\sqrt{n}}\right) \text{ a.s.},$$

This means

$$\frac{[X_{(k_2 n)} - X_{(k_1 n)}]^2}{4z_{\alpha/2}^2} = \frac{p(1-p)}{n[F'(\theta)]^2} + o\left(\frac{1}{n}\right) \text{ a.s.}$$

Therefore, $[X_{(k_2 n)} - X_{(k_1 n)}]^2/(4z_{\alpha/2}^2)$ is a consistent estimator of the asymptotic variance of the sample $p$th quantile.
From Theorem 5.10, if $F'(\theta)$ exists and is positive, then
\[
\sqrt{n}(\hat{\theta}_n - \theta) \rightarrow_d N\left(0, \frac{p(1-p)}{[F'(\theta)]^2}\right).
\]

If the derivative $F'(\theta)$ has a consistent estimator $\hat{d}_n$ obtained using some method such as one of those introduced in §5.1.3, then $\hat{V}_n = p(1-p)/\hat{d}_n^2$ is a consistent estimator of $p(1-p)/[F'(\theta)]^2$ and the method introduced in §7.3.1 can be applied to derive the following $1 - \alpha$ asymptotically correct confidence interval:

\[
C_1(X) = \left[\hat{\theta}_n - z_{1-\alpha/2} \frac{\sqrt{p(1-p)}}{\hat{d}_n \sqrt{n}}, \hat{\theta}_n + z_{1-\alpha/2} \frac{\sqrt{p(1-p)}}{\hat{d}_n \sqrt{n}}\right].
\]

The length of $C_1(X)$ is asymptotically almost the same as Woodruff’s interval. However, $C_1(X)$ depends on the estimated derivative $\hat{d}_n$ and it is usually difficult to obtain a precise estimator $\hat{d}_n$. 