# CS 540 Introduction to Artificial Intelligence
# **Games II**

Fred Sala
University of Wisconsin-Madison

**April 6, 2021**

# Announcements

- **Homeworks**:
  - HW8: Game AI released. Use this set of slides for reference.
- **Midterm**: grading nearly done.
- Class roadmap:

| | |
|---|---|
| Thursday, April 1 | Games I |
| **Tuesday, April 6** | **Games II** |
| Thursday, April 8 | Search I |
| Tuesday, April 13 | Search II |

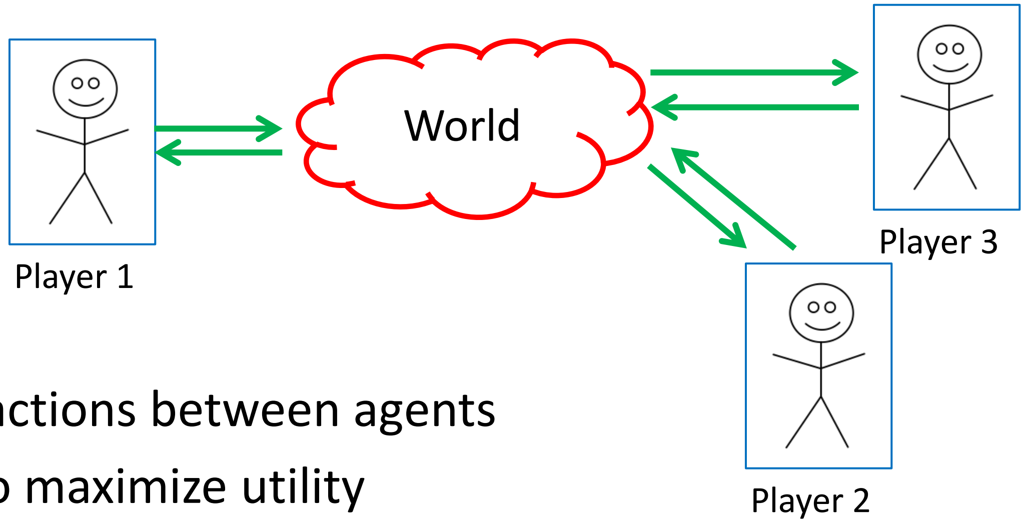
Artificial Intelligence

# Outline

- Review of game theory basics
  - Properties, mathematical setup, simultaneous games
- Sequential games
  - Game trees, minimax, search approaches
- Speeding up sequential game search
  - Pruning, heuristics

# Review of Games: Multiple Agents

Games setup: **multiple** agents




- Now: interactions between agents
- Still want to maximize utility
- **Strategic** decision making.

# Review of Games: Properties

Let's work through **properties** of games

- **Number** of agents/players
- State & action spaces: **discrete** or **continuous**
- **Finite** or **infinite**
- **Deterministic** or **random**
- **Sum**: zero or positive or negative
- **Sequential** or **simultaneous**

Wiki

# Review: Prisoner's Dilemma

**Famous** example from the '50s.

Two prisoners A & B. Can choose to betray the other or not.

- A and B both betray, each of them serves two years in prison
- One betrays, the other doesn't: betrayer free, other three years
- Both do not betray: one year each

Properties: **2-player, discrete, finite, deterministic, negative-sum, simultaneous**

# Review: Normal Form

Mathematical description of simult. games. Has:

- $n$ players $\{1,2,\ldots,n\}$
- Player $i$ strategy $a_i$ from $A_i$. **All**: $a = (a_1, a_2, \ldots, a_n)$
- Player $i$ gets rewards $u_i(a)$ for any outcome
  - **Note**: reward depends on other players!

- Setting: all of these spaces, rewards are **known**

# Review: Example of Normal Form

**Ex**: Prisoner's Dilemma

| Player 2 / Player 1 | *Stay silent* | *Betray* |
|---|---|---|
| *Stay silent* | −1, −1 | −3, 0 |
| *Betray* | 0, −3 | −2, −2 |

- 2 players, 2 actions: yields 2x2 matrix
- Strategies: {Stay silent, betray} (i.e, binary)
- Rewards: {0,-1,-2,-3}

# Review: Dominant Strategies

Let's analyze such games. Some strategies are better

- Dominant strategy: if $a_i$ better than $a_i'$ *regardless* of what other players do, $a_i$ is **dominant**
- I.e.,

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \neq a_i \text{ and } \forall a_{-i}$$

All of the other entries
of *a* excluding *i*

- Doesn't always exist!

# Review: Equilibrium

$a*$ is an equilibrium if all the players do not have an incentive to **unilaterally deviate**

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i$$

- All players dominant strategies -> equilibrium
- Converse doesn't hold (don't need dominant strategies to get an equilibrium)

# Review: Pure and Mixed Strategies

So far, all our strategies are deterministic: "**pure**"

- Take a particular action, no randomness

Can also randomize actions: "**mixed**"

- Assign probabilities $x_i$ to each action

$$x_i(a_i), \text{ where } \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0$$

- Note: have to now consider **expected rewards**

# Review: Nash Equilibrium

Consider the mixed strategy $x^*$ = ($x_1^*$, …, $x_n^*$)

- This is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-1}^*) \geq u_i(x_i, x_{-i}^*) \quad \forall x_i \in \Delta_{A_i}, \forall i \in \{1, \ldots, n\}$$

Better than doing anything else, "**best response**"

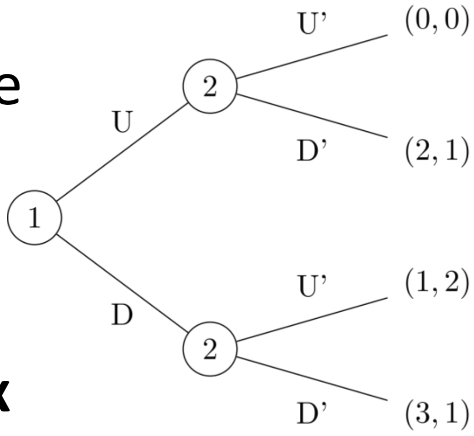
Space of probability distributions

- Intuition: nobody can **increase expected reward** by changing only their own strategy. A type of solution!

# Sequential Games

More complex games with multiple moves

- Instead of normal form, **extensive form**
- Represent with a **tree**
- Find strategies: perform search over the tree




Wiki

- Can still look for Nash equilibrium
  - Or, other criteria like **maximin** / **minimax**

# II-Nim: Example Sequential Game

2 piles of sticks, each with 2 sticks.

- Each player takes one or more sticks from pile
- Take last stick: lose

(ii, ii)

- Two players: Max and Min
- If Max wins, the score is **+1**; otherwise **-1**
- Min's score is –Max's
- Use Max's as the score of the game

# Game Trajectory

(ii, ii)

# Game Trajectory

(ii, ii)

Max takes one stick from one pile

(i, ii)

# Game Trajectory

(ii, ii)

Max takes one stick from one pile

(i, ii)

Min takes two sticks from the other pile

(i,-)

# Game Trajectory

(ii, ii)

Max takes one stick from one pile

(i, ii)

Min takes two sticks from the other pile

(i,-)

Max takes the last stick

(-,-)

Max gets score **-1**

# Game tree for II-Nim

Two players:
Max and Min

(ii ii) Max

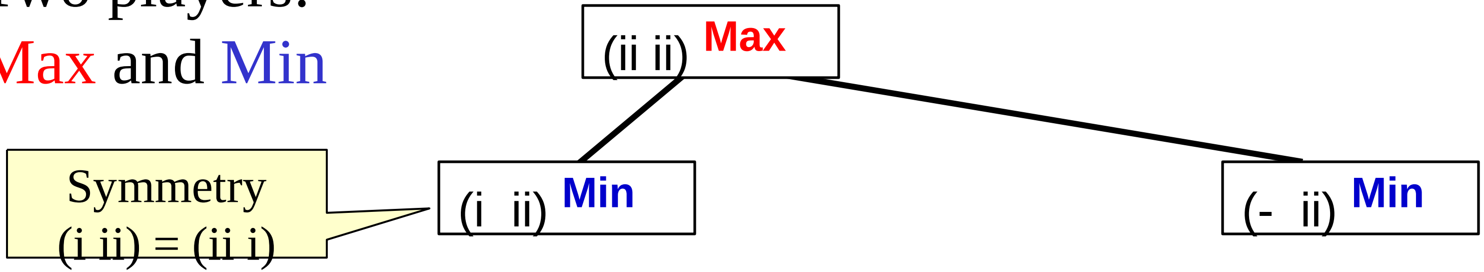
who is to move at this state

Convention: score is w.r.t. the first player Max. Min's score = – Max

Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

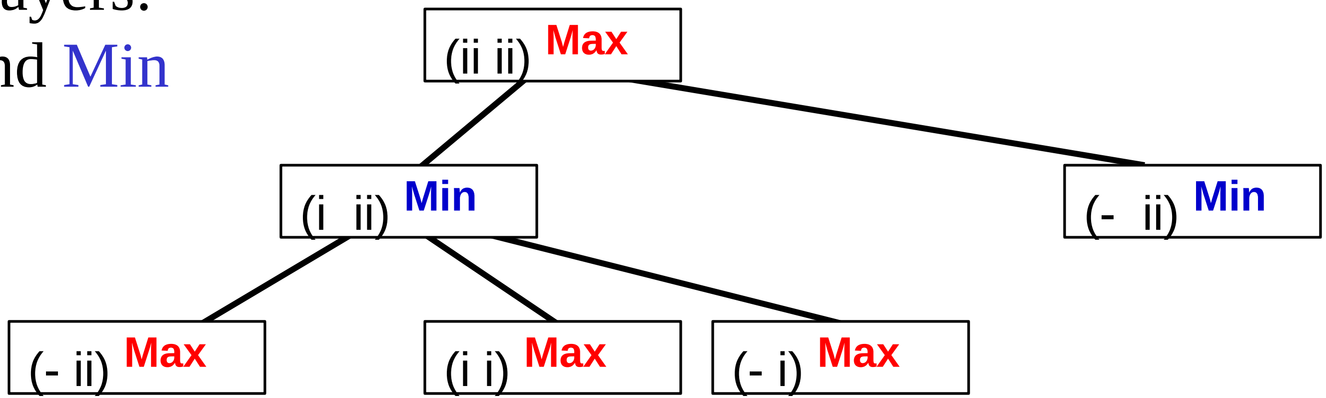
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

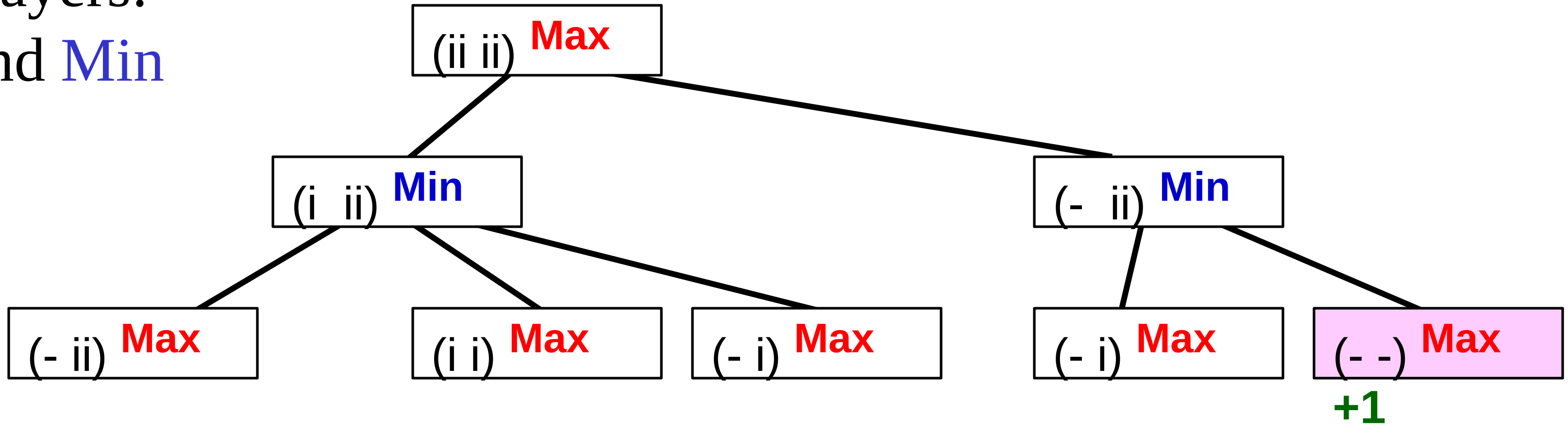
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

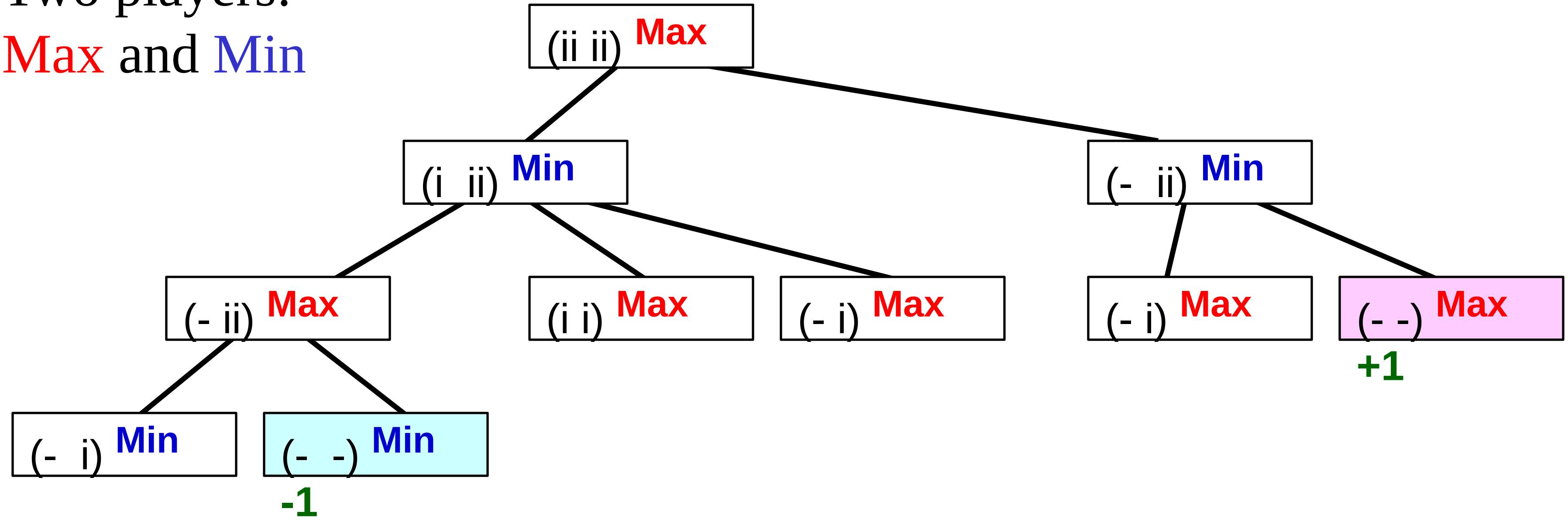
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

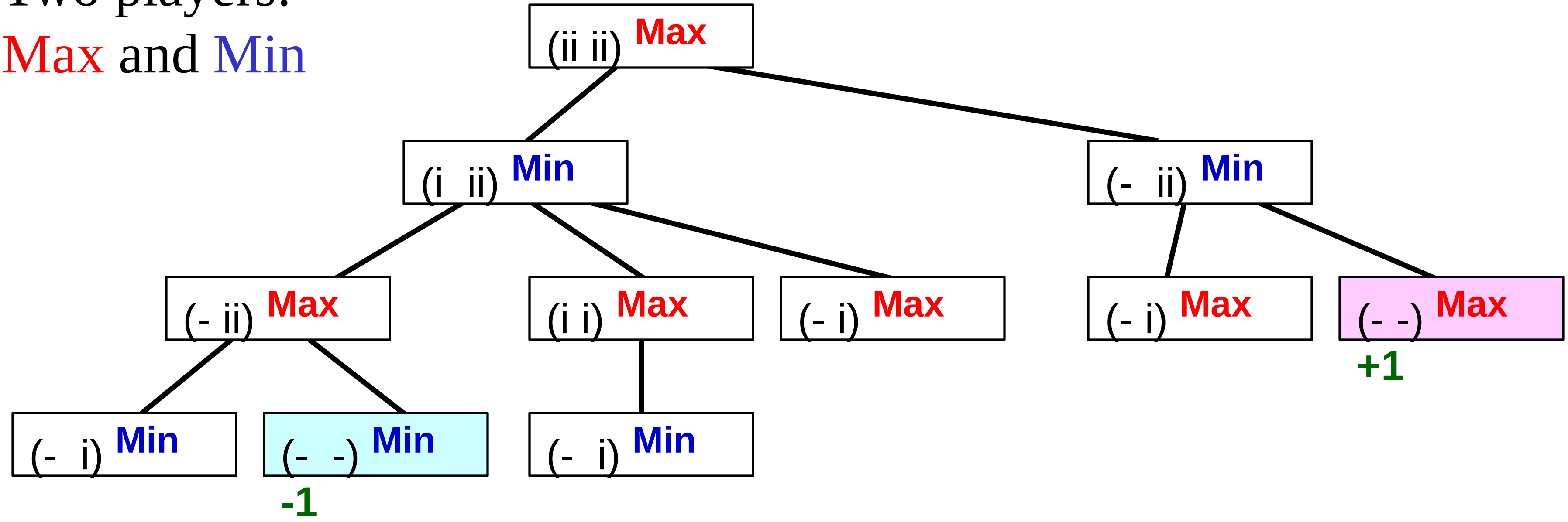
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

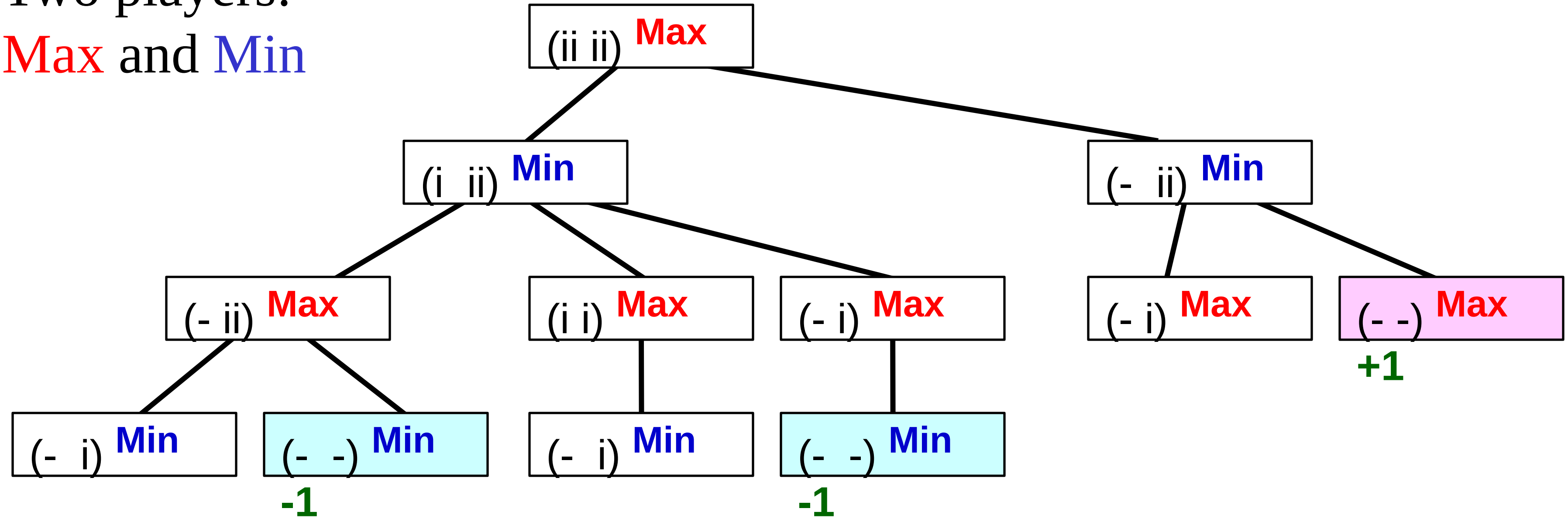
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

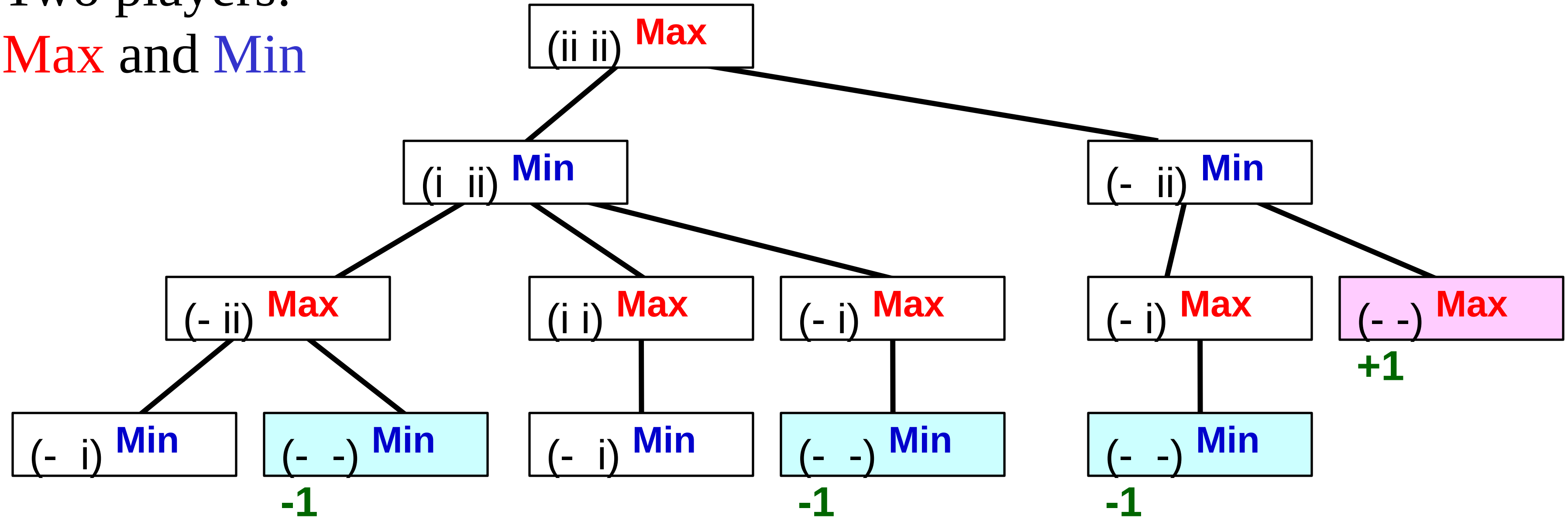
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

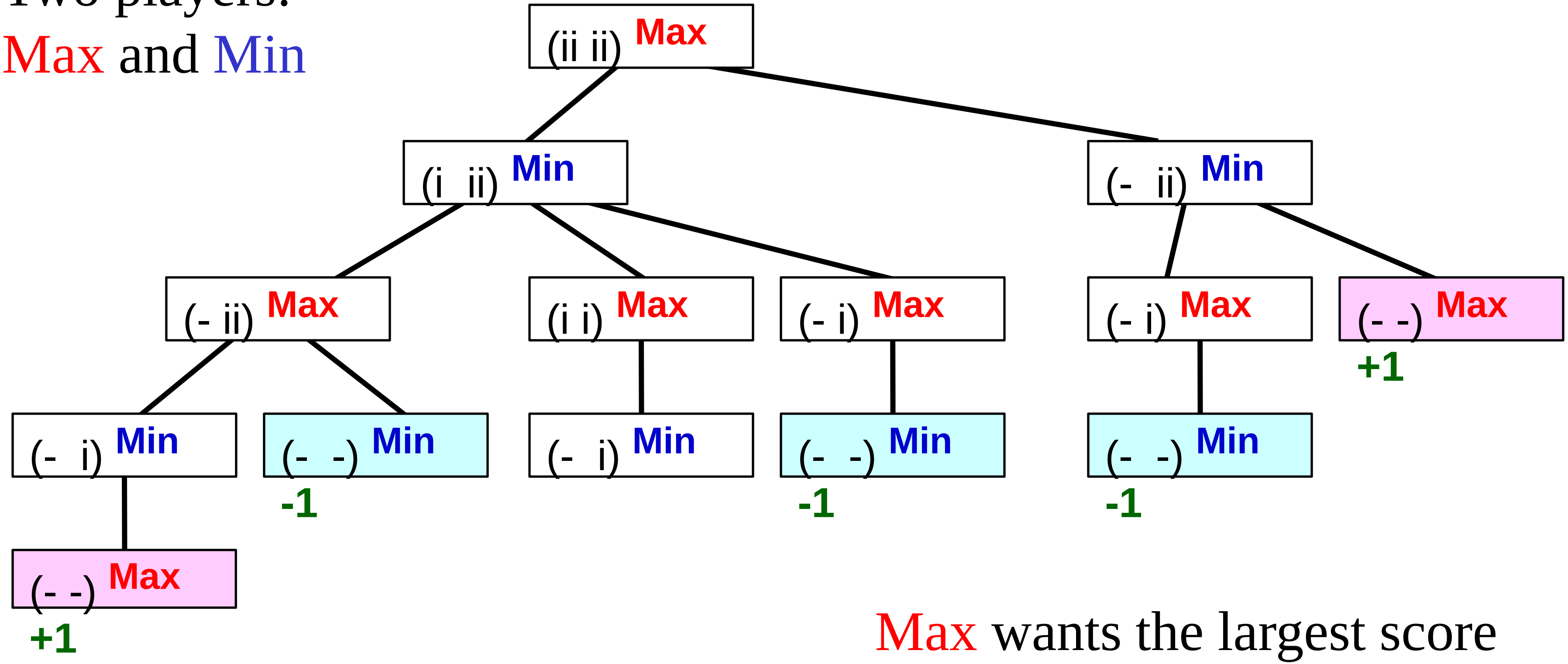
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

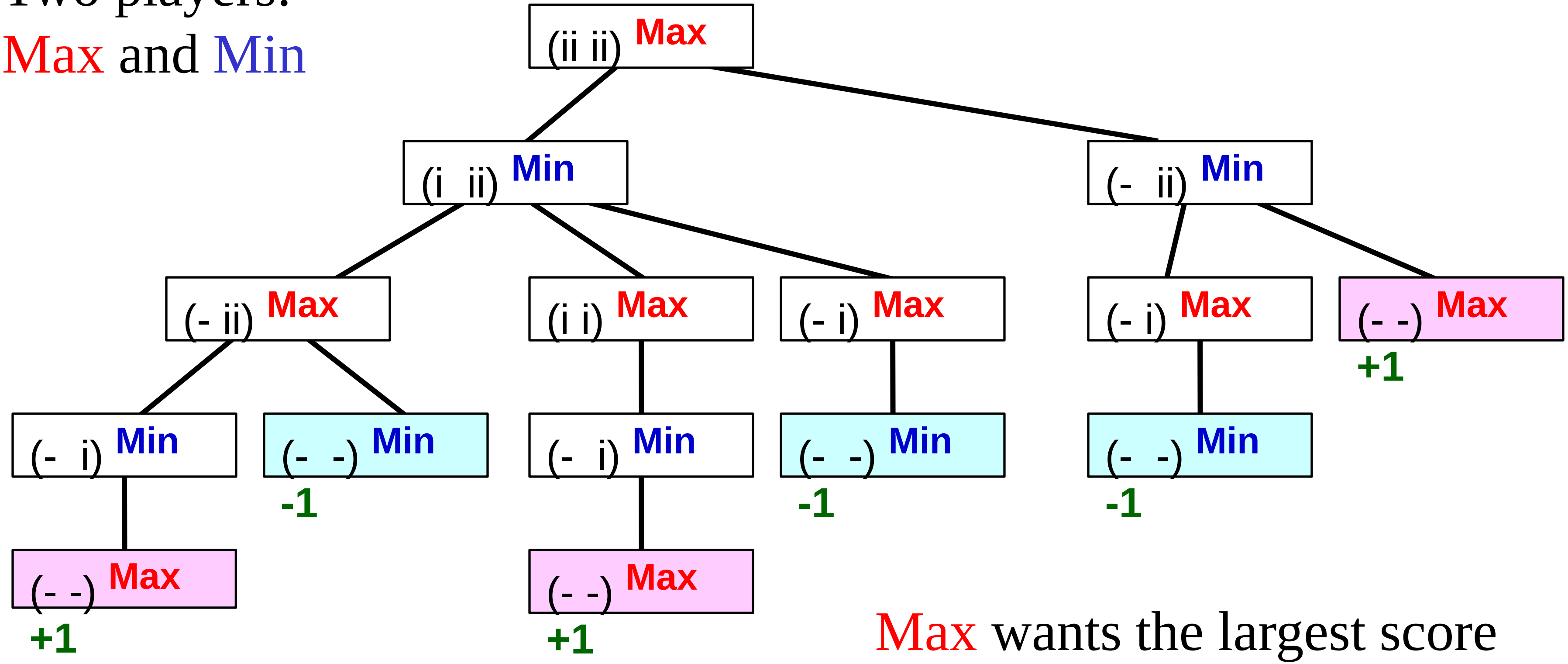
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Strategies & Rewards

Let’s stick to zero-sum two-player games

- Strategies: player 1 (Max): *s*, player 2 (Min): *t*
- Player 1 (Max): reward $u(s,t)$, player 2 (Min): $-u(s,t)$

- Max goal: maximize $u(s,t)$

- Goal: find strategies *s*, *t* that do this.

# Minimax Theorem

Famous result of von Neumann

- Says: there are strategies $s^*$ and $t^*$ and a value $u^*$, the **minimax value** so that
  - If **Min** uses $t^*$, then **Max's** reward $\leq u^*$ (i.e., $\max_s u(s, t^*) = u^*$)
  - If **Max** uses $s^*$, then **Max's** reward $\geq u^*$ (i.e., $\min_t u(s^*, t) = u^*$)
- So: $\mathrm{u}(s^*, t^*) = u^*$
- Also: if game has perfect information, there are pure strategies $s^*$, $t^*$ that satisfy the result

# Finding The Strategies

Back to our game tree

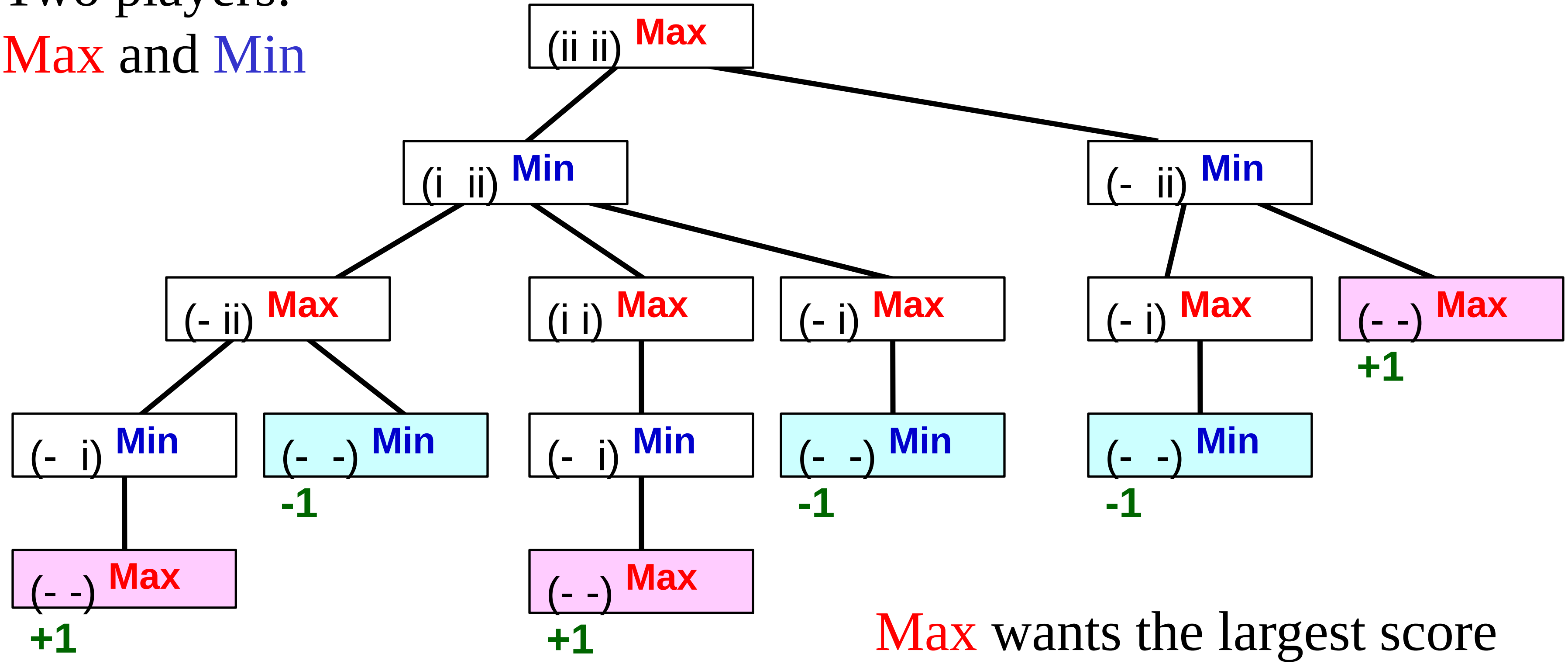
- Write down all the pure strategies (e.g., the big tree) and select the *s** and *t**

$$s^* = \arg\max_{s \in S} \min_{t \in T} u(s,t) \qquad t^* = \arg\min_{t \in T} \max_{s \in S} u(s,t)$$

- Big search, since for branching factor *b*, height *h*, need to look at ~$b^h$ strategies

# Game tree for II-Nim

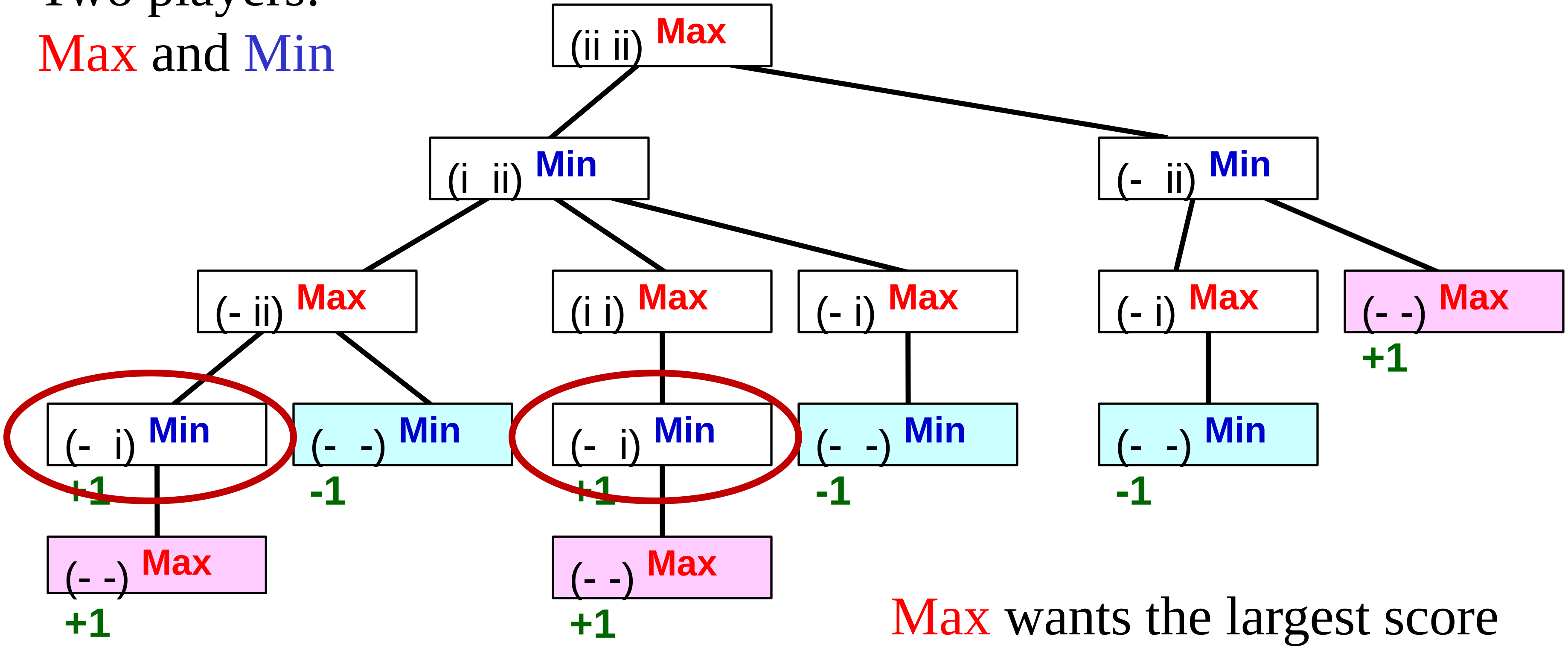
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

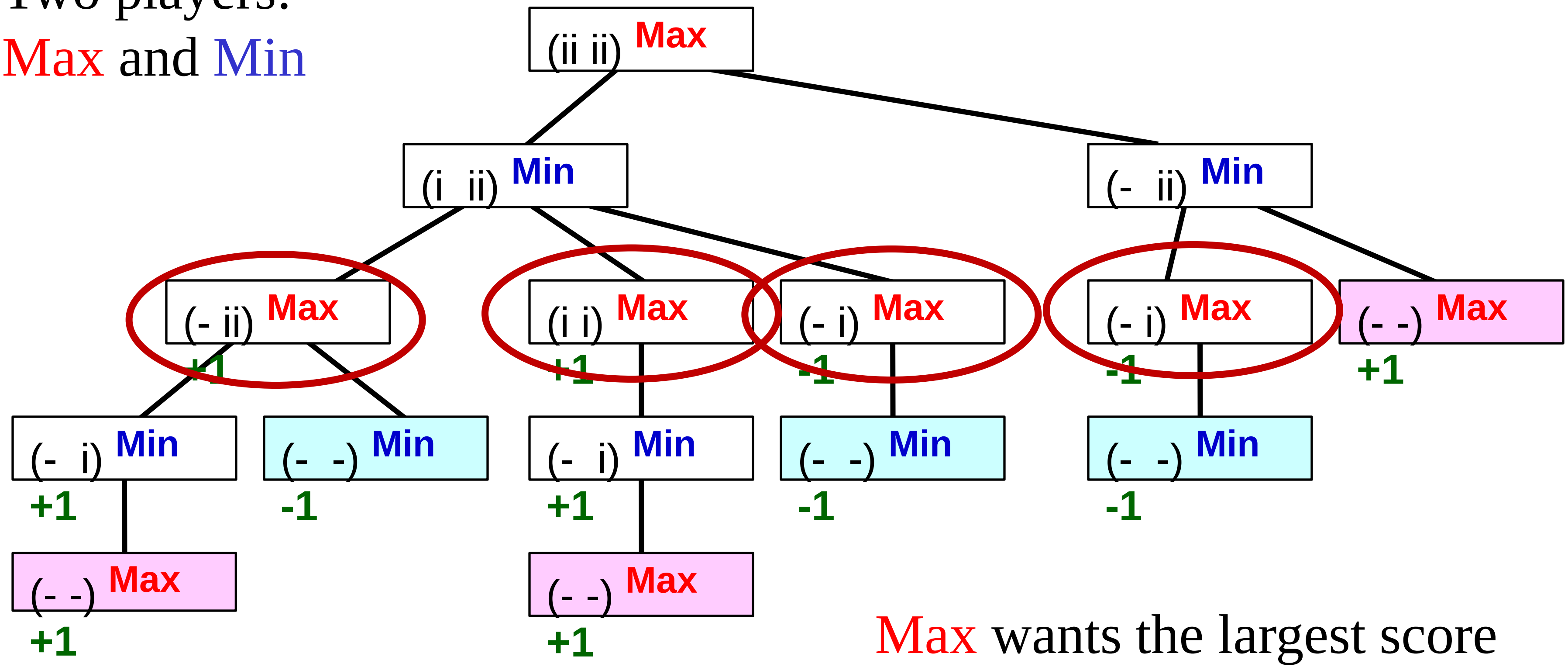
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

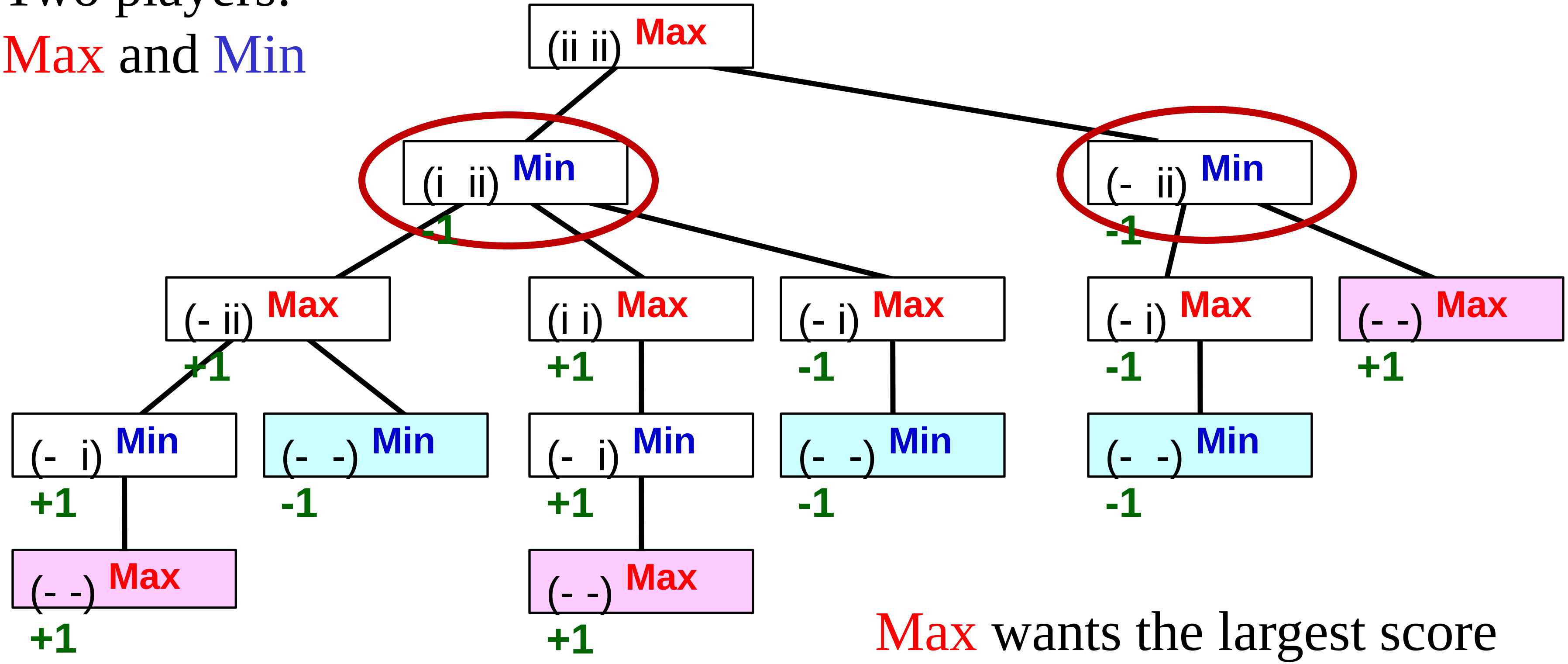
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

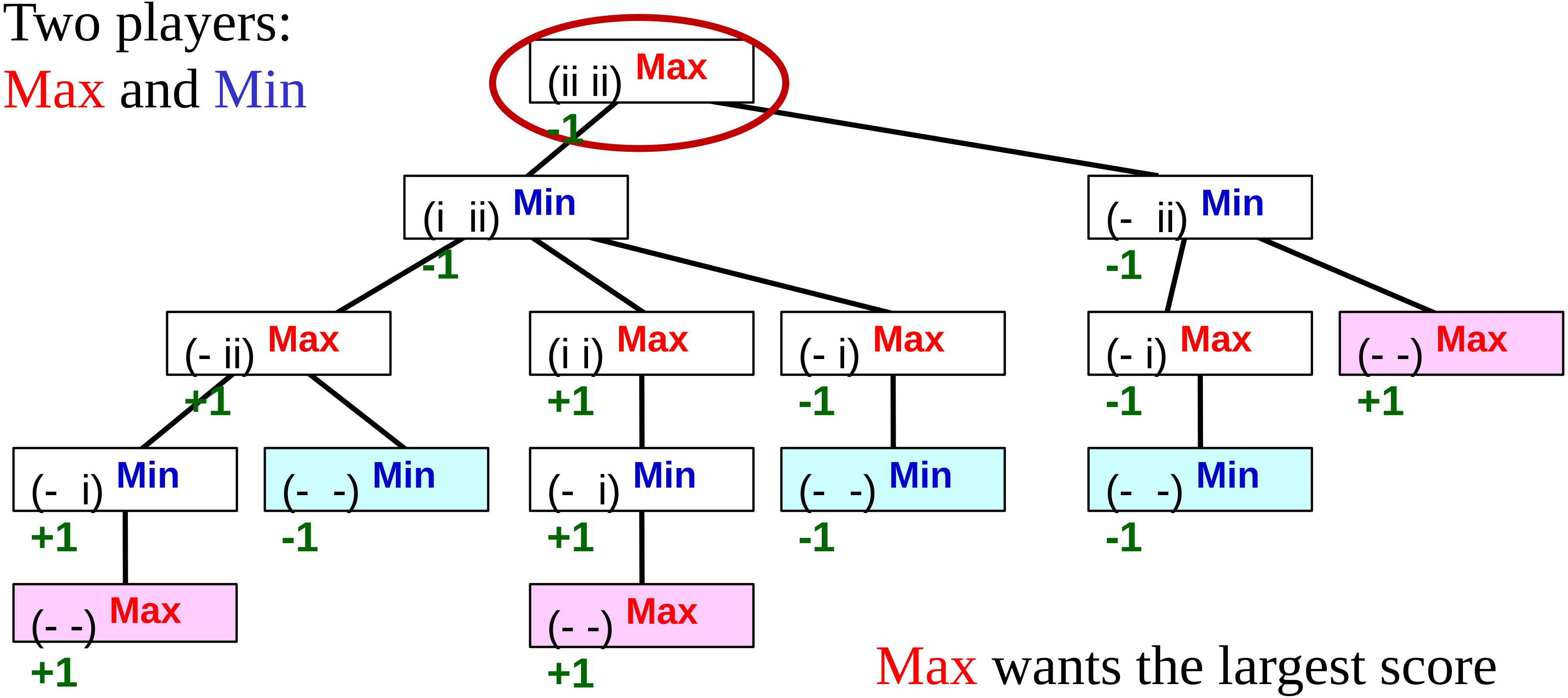
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

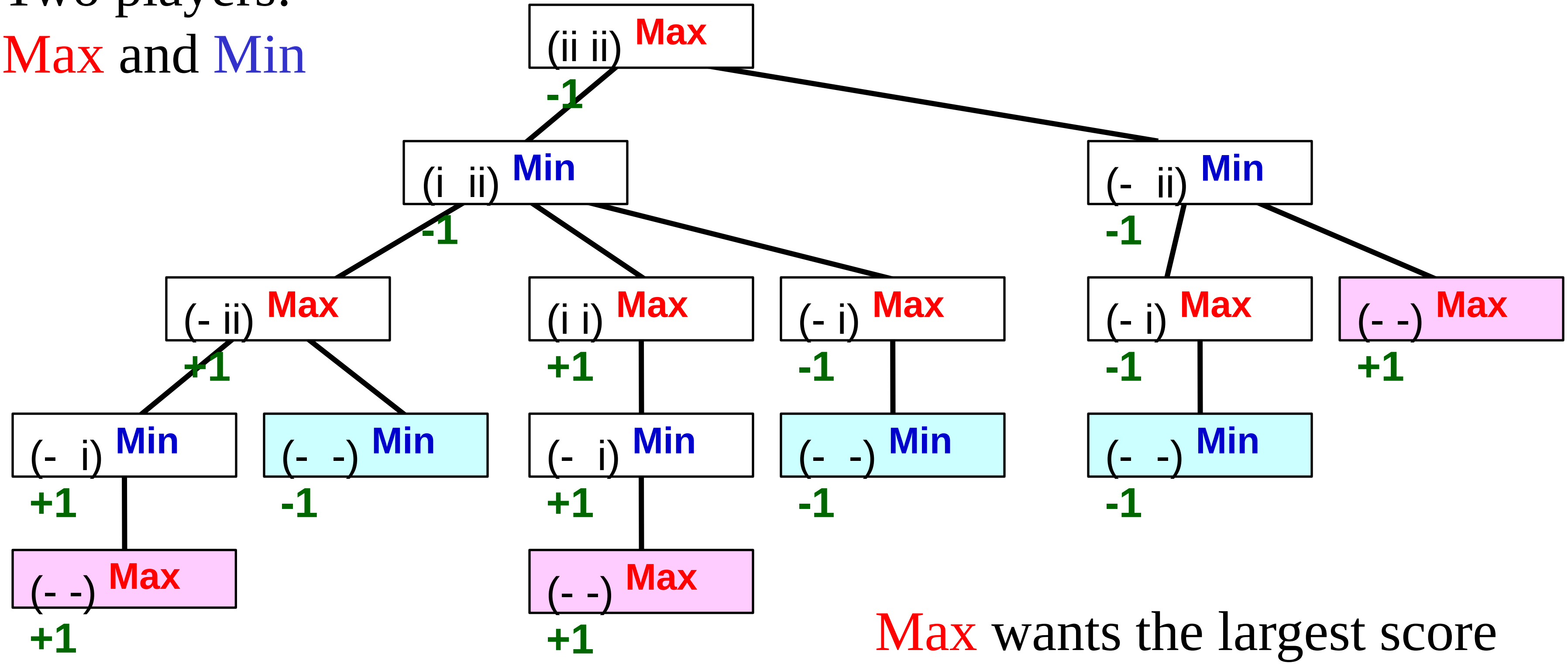
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

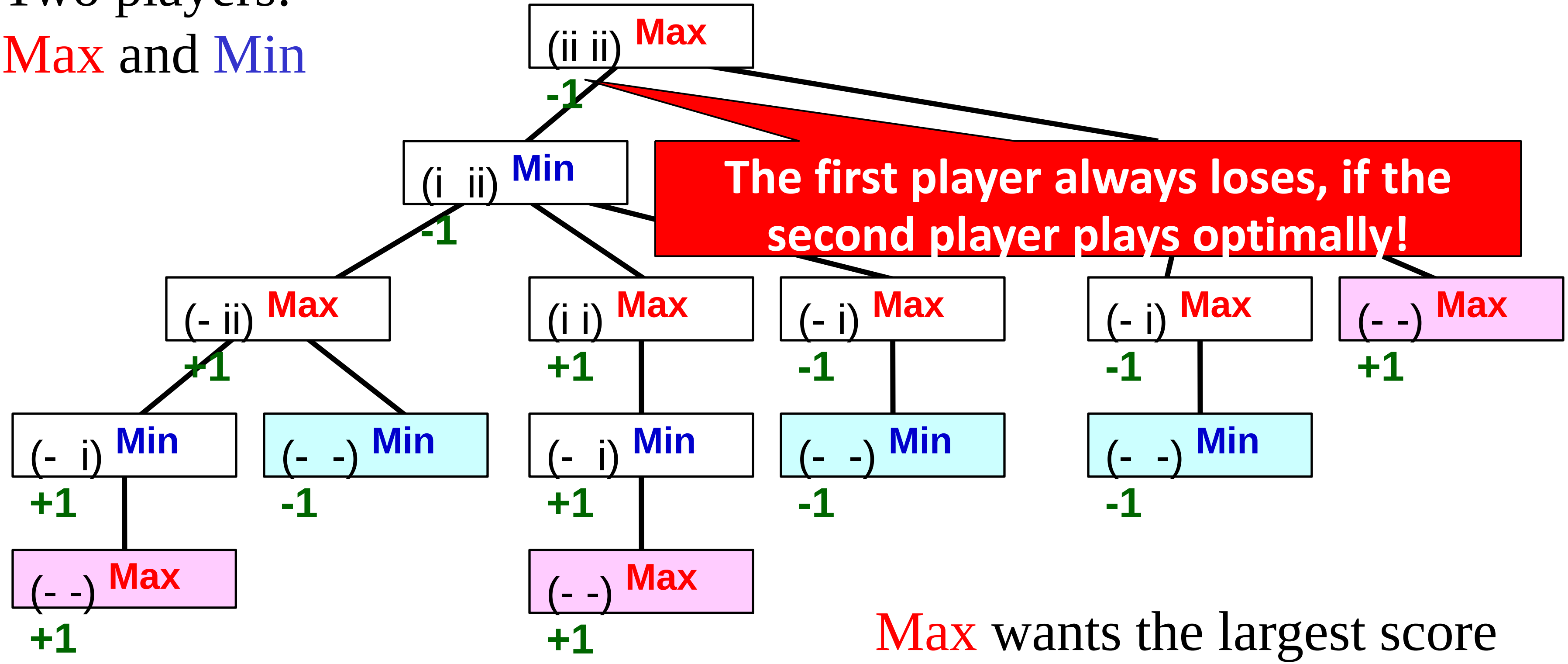
Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Game tree for II-Nim

Two players:
Max and Min




Max wants the largest score
Min wants the smallest score

# Our Approach So Far

We find the minimax value/strategy bottom up

- Minimax value: score of terminal node when both players play optimally
  - **Max's** turn, take max of children
  - **Min's** turn, take min of children

- Can implement this as depth-first search: **minimax algorithm**

# Minimax Algorithm

```
function Max-Value(s)
inputs:
      s: current state in game, Max about to play
output: best-score (for Max) available from s

      if ( s is a terminal state )
      then return ( terminal value of s )
      else
                  α := – infinity
                  for each s' in Succ(s)
                     α := max( α , Min-value(s'))
      return α
```

```
function Min-Value(s)
output: best-score (for Min) available from s

      if ( s is a terminal state )
      then return ( terminal value of s)
      else
                  β := infinity
                  for each s' in Succs(s)
                     β := min( β , Max-value(s'))
      return β
```

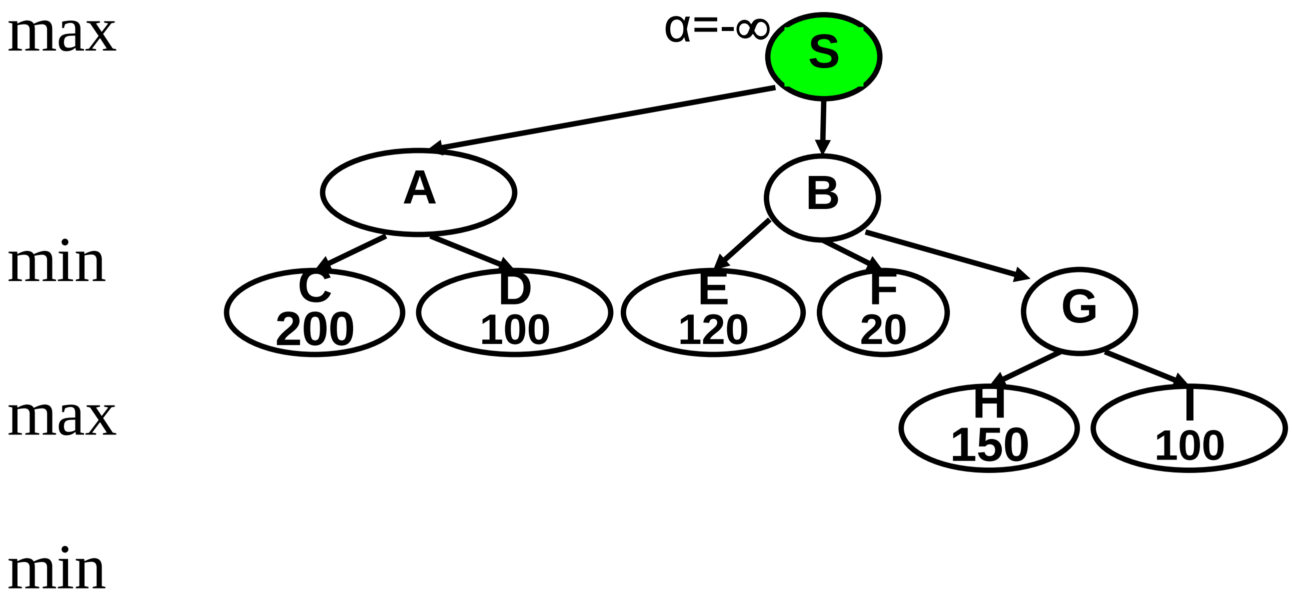
Time complexity?

- $O(b^m)$

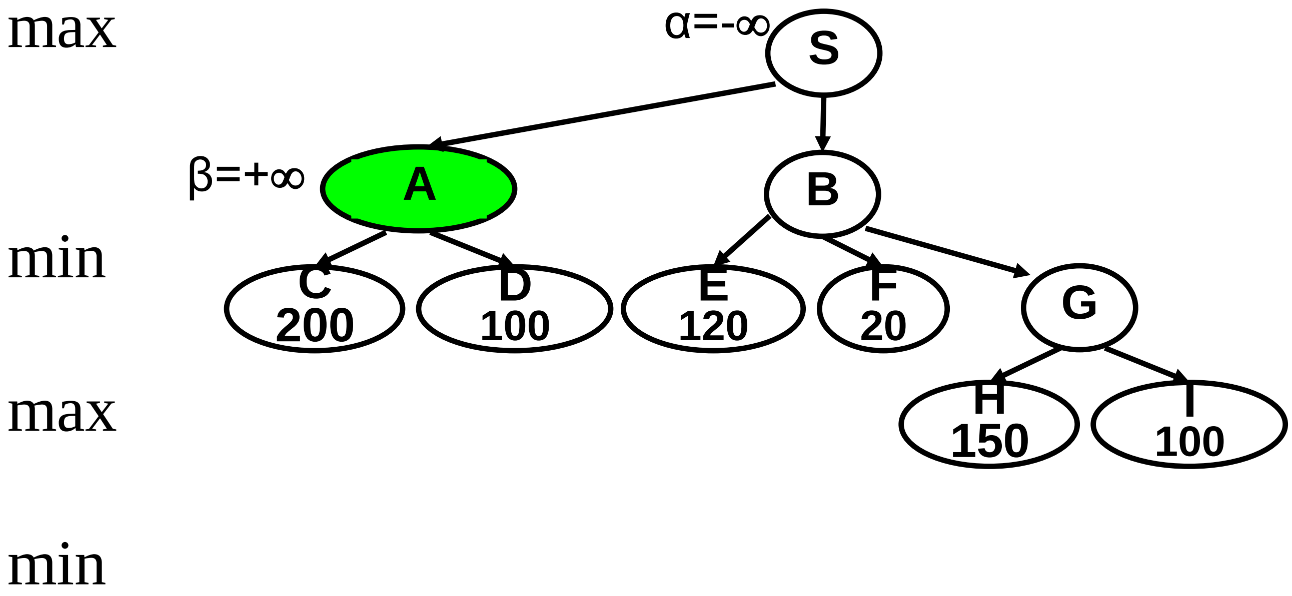
Space complexity?

- $O(bm)$

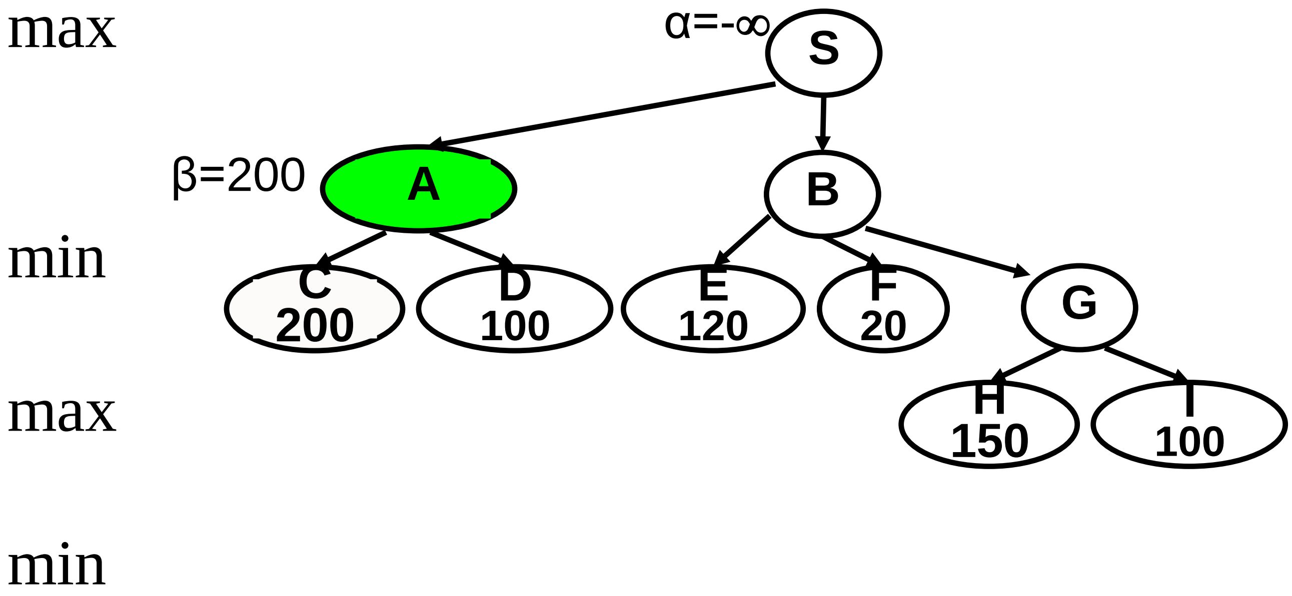
# Minimax algorithm in execution



# Minimax algorithm in execution

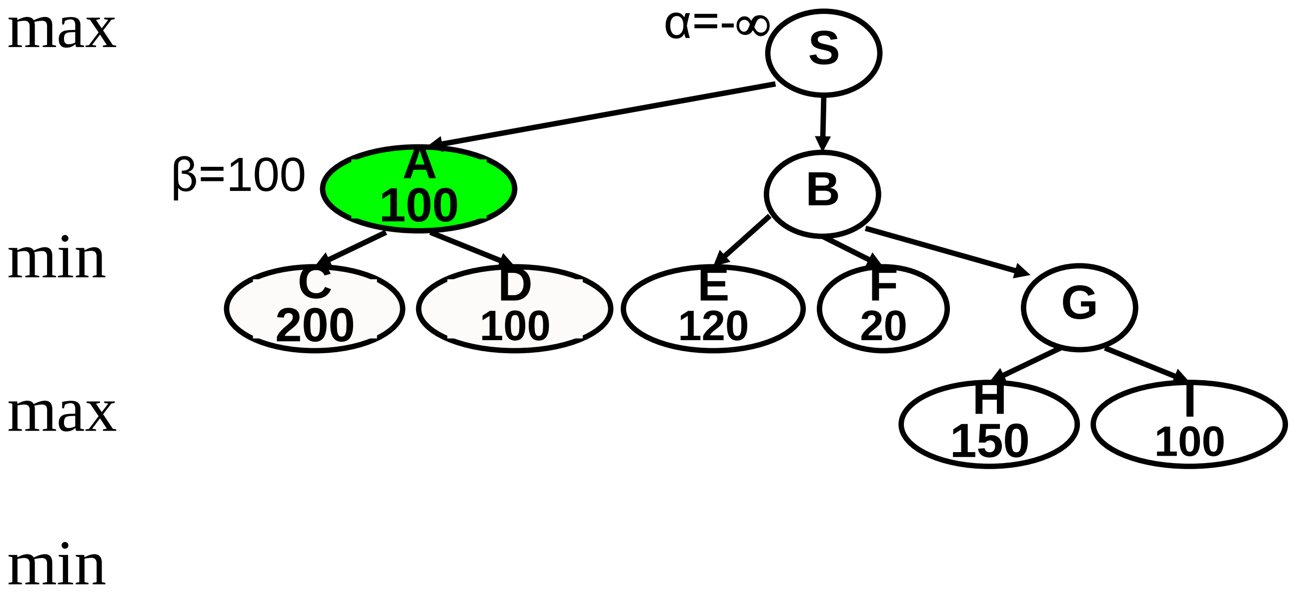


# Minimax algorithm in execution

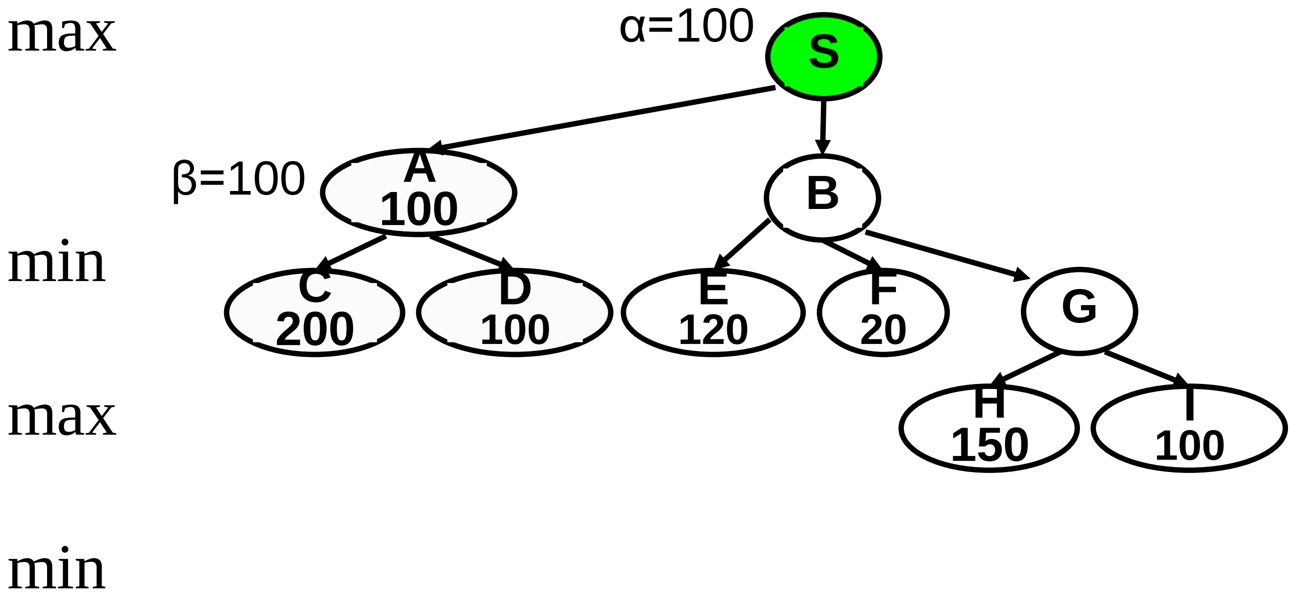



The execution on the terminal nodes is omitted.

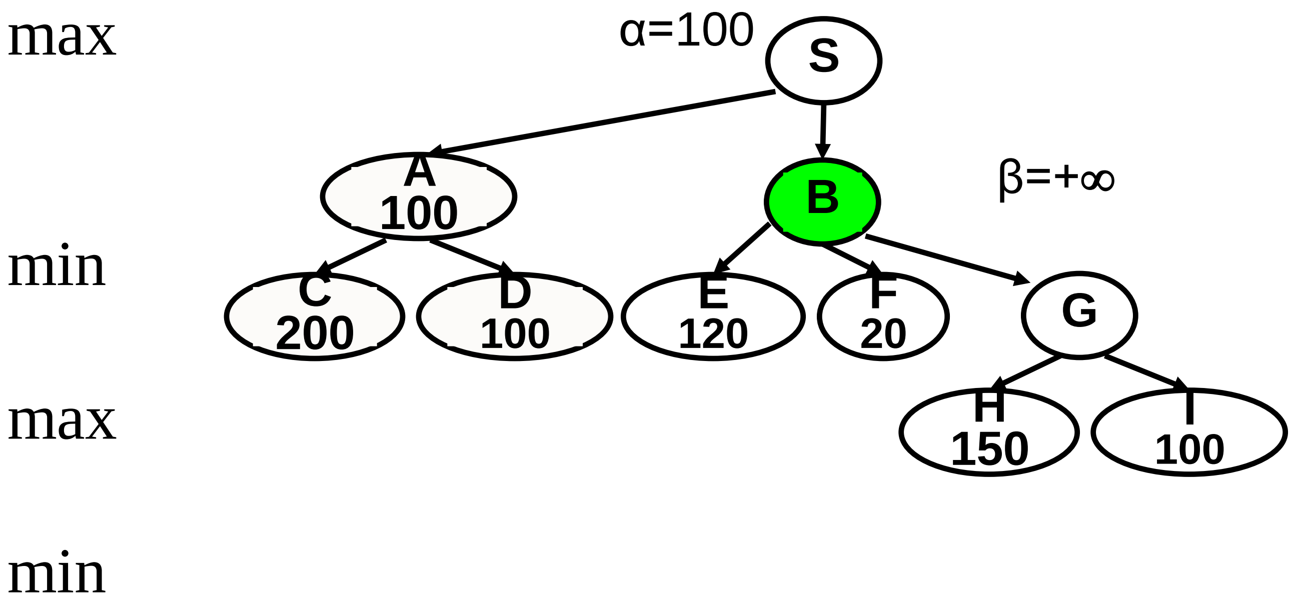
# Minimax algorithm in execution



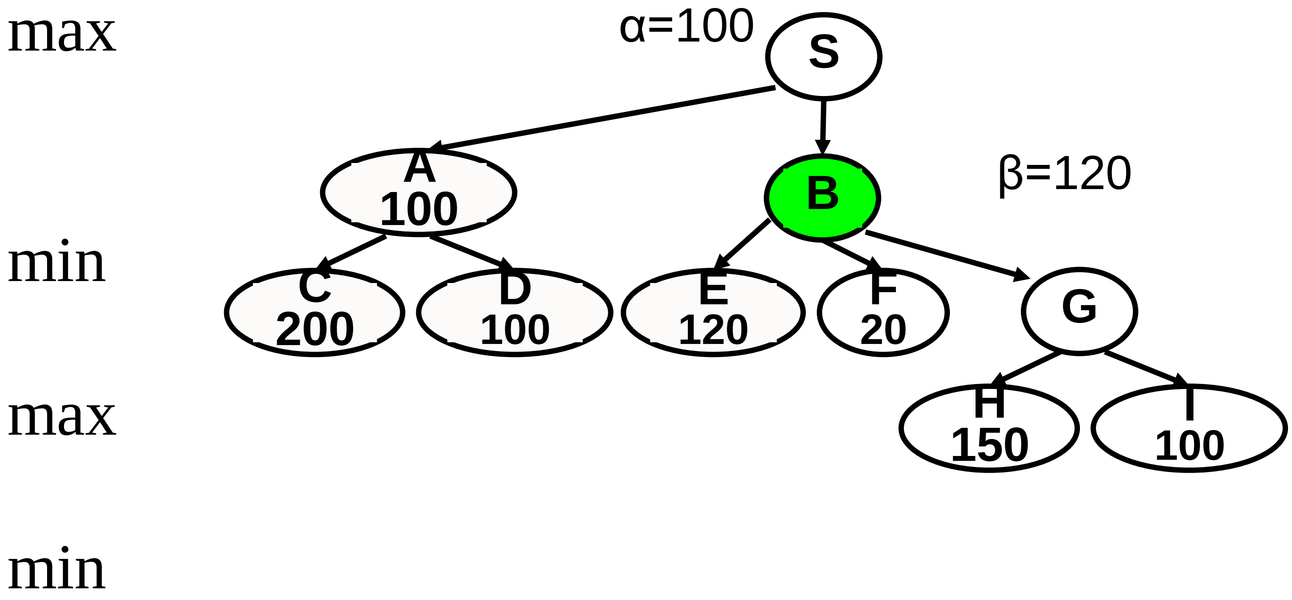
# Minimax algorithm in execution



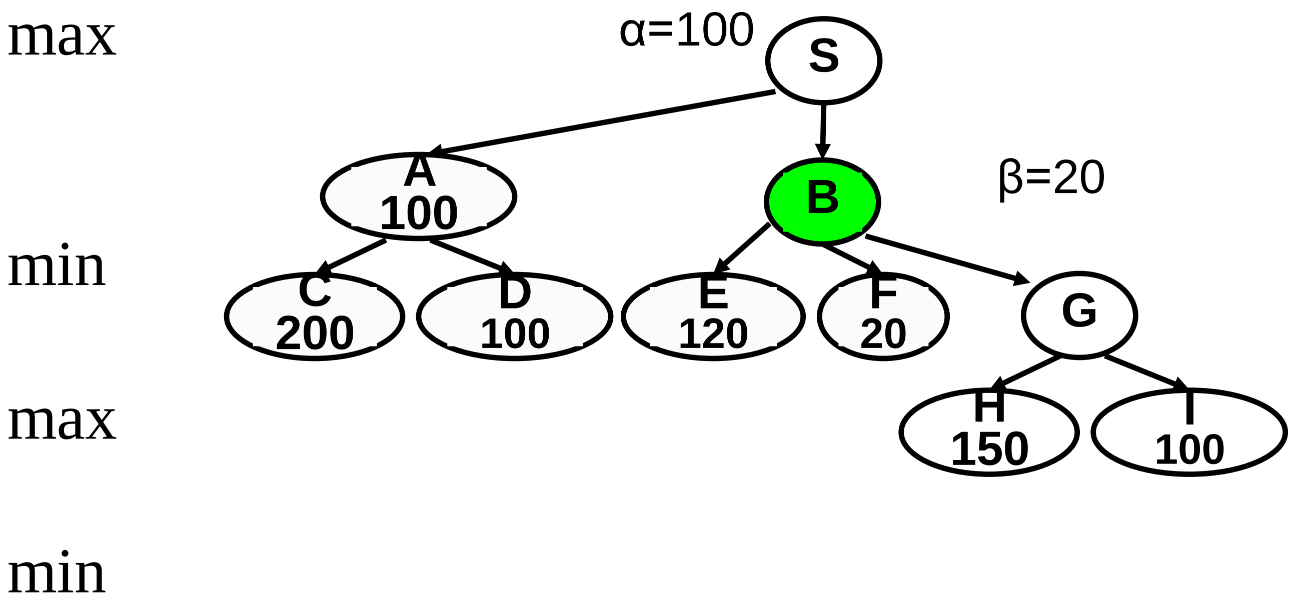
# Minimax algorithm in execution



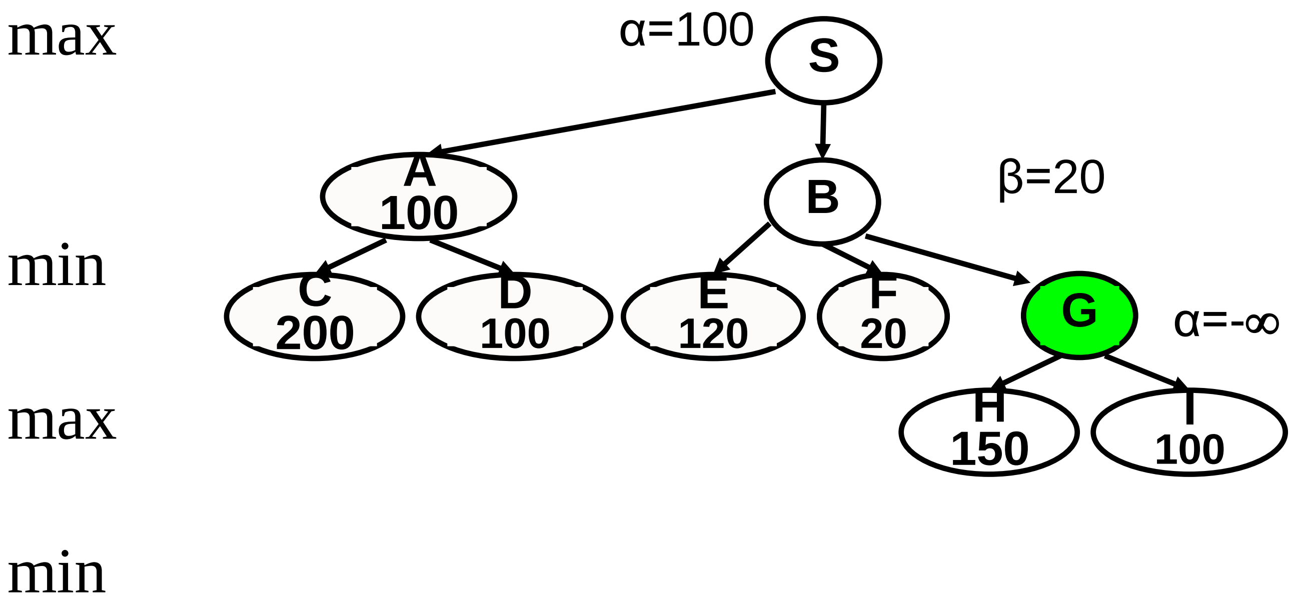
# Minimax algorithm in execution



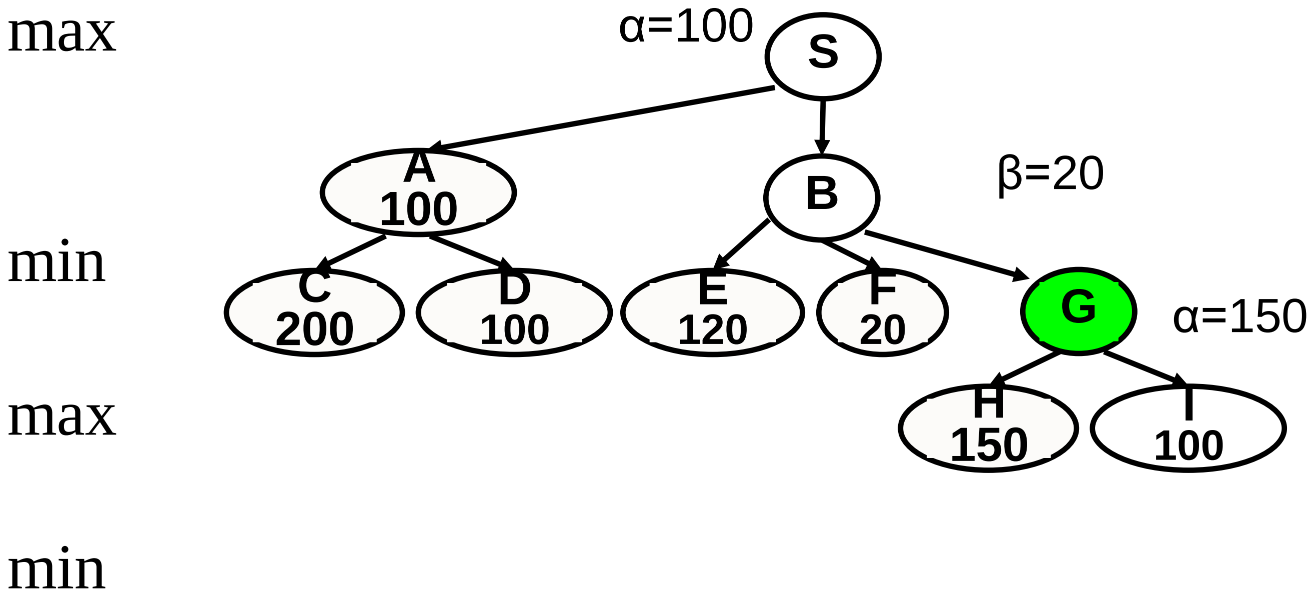
# Minimax algorithm in execution



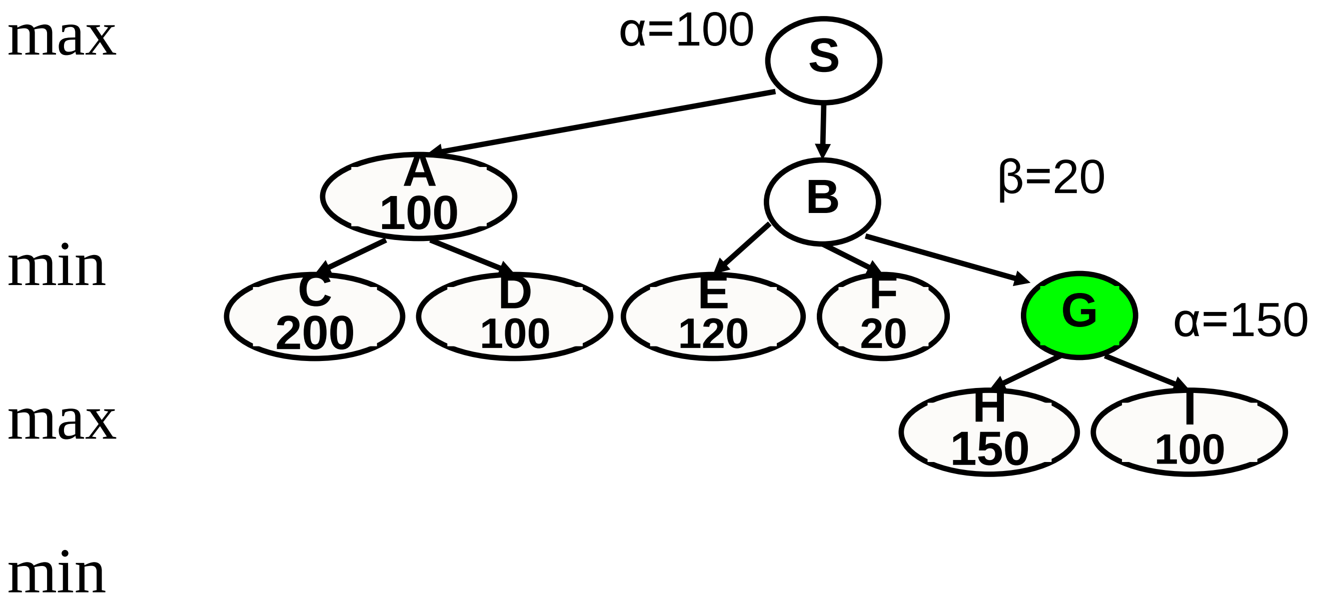
# Minimax algorithm in execution



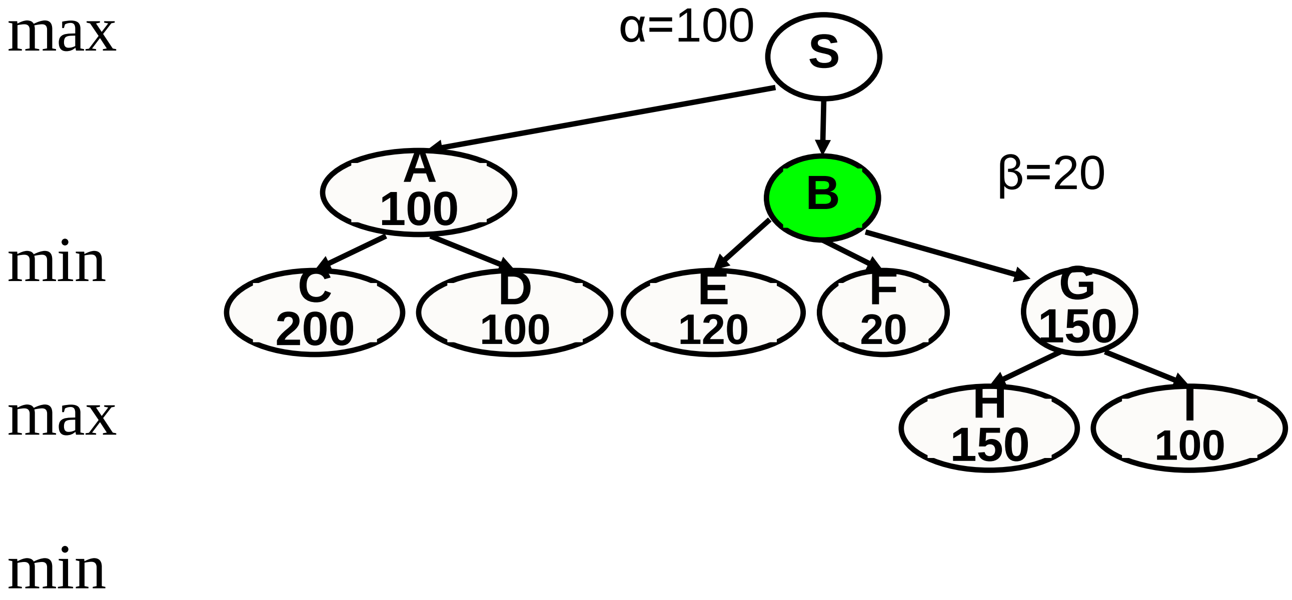
# Minimax algorithm in execution



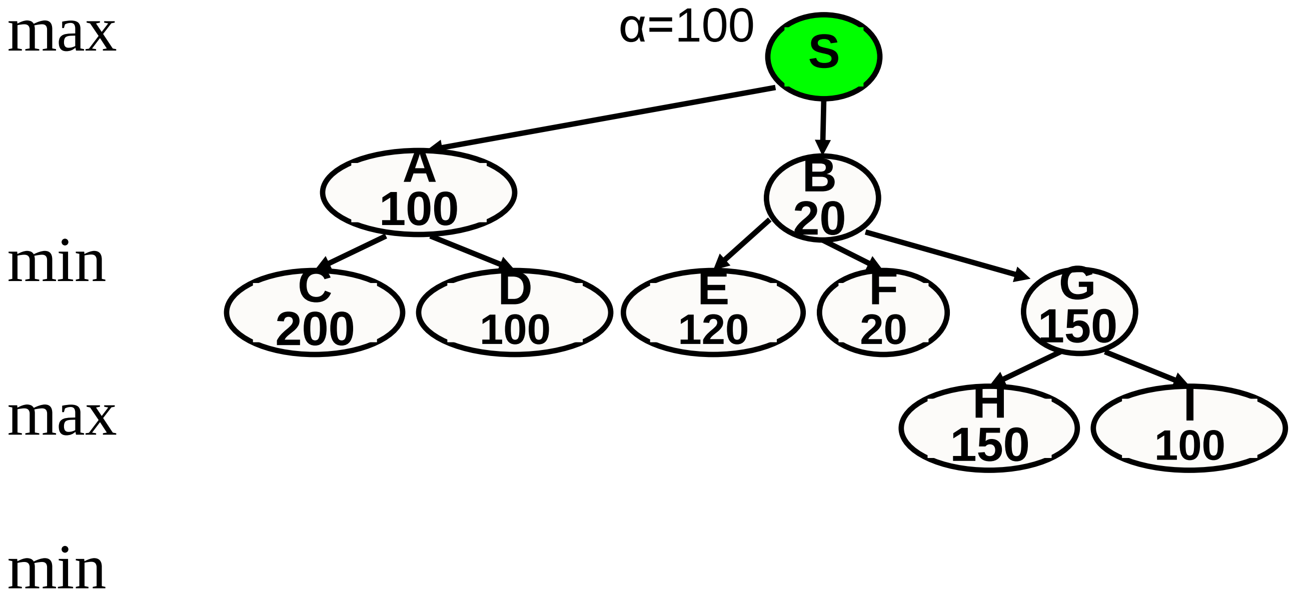
# Minimax algorithm in execution



# Minimax algorithm in execution



# Minimax algorithm in execution



# Can We Do Better?

One **downside**: we had to examine the entire tree

An idea to speed things up: **pruning**

- Goal: want the same minimax value, but faster
- We can get rid of bad branches

# Alpha-beta pruning

```
function Max-Value (s,α,β)
inputs:
      s: current state in game, Max about to play
      α: best score (highest) for Max along path to s
      β: best score (lowest) for Min along path to s
output: min(β , best-score (for Max) available from s)

      if ( s is a terminal state )
      then return ( terminal value of s )
      else for each s’ in Succ(s)
        α := max( α , Min-value(s’,α,β))
        if ( α ≥ β ) then return β   /* alpha pruning */
      return α
```

```
function Min-Value(s,α,β)
output: max(α , best-score (for Min) available from s )

      if ( s is a terminal state )
      then return ( terminal value of s)
      else for each s’ in Succs(s)
        β := min( β , Max-value(s’,α,β))
      if (α ≥ β ) then return α   /* beta pruning */
      return β
```

Starting from the root:
Max-Value(root, -∞, +∞)

# Alpha-Beta Pruning

## How effective is **alpha-beta pruning**?

- Depends on the order of successors!
  - Best case, the #of nodes to search is O($b^{m/2}$)
  - Happens when each player's best move is the leftmost child.
  - The worst case is no pruning at all.

- In DeepBlue, the average branching factor was about 6 with alpha-beta instead of 35-40 without.

# Minimax With Heuristics

Note that long games are yield huge computation

- To deal with this: limit ***d*** for the search depth
- **Q**: What to do at depth *d*, but no termination yet?
  - **A**: Use a heuristic evaluation function *e(x)*

**function** MINIMAX($x, d$) **returns** an estimate of $x$'s utility value
  **inputs**: $x$, current state in game
    $d$, an upper bound on the search depth
  **if** $x$ is a terminal state **then return** Max's payoff at $x$
  **else if** $d = 0$ **then return** $e(x)$
  **else if** it is Max's move at $x$ **then**
    **return** $\max\{\text{MINIMAX}(y, d-1) : y \text{ is a child of } x\}$
  **else return** $\min\{\text{MINIMAX}(y, d-1) : y \text{ is a child of } x\}$

Credit: Dana Nau

# Heuristic Evaluation Functions

- e(x) often a weighted sum of features (like our linear models)

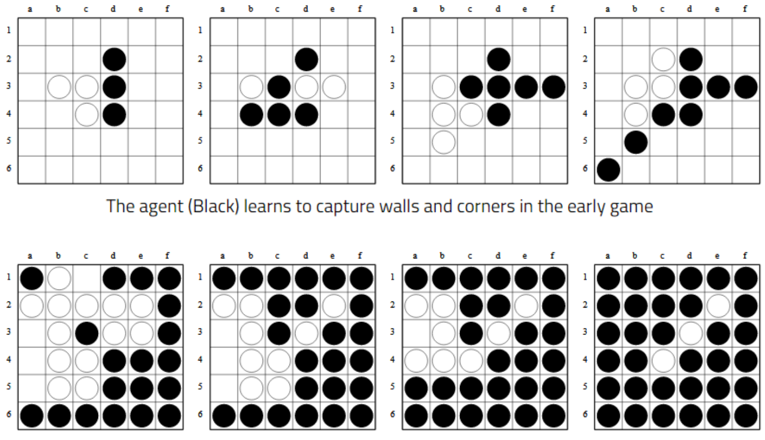
$$e(x) = w_1 f_1(x) + w_2 f_2(x) + \ldots + w_n f_n(x)$$

- Chess example: $f_i(x)$ = **difference** between number of white and black, with *i* ranging over piece types.
  - Set weights according to piece importance
  - E.g., 1(# white pawns - # black pawns) + 3(#white knights - # black knights)

# Going Further

- Monte Carlo tree search (MCTS)
  - Uses random sampling of the search space
  - Choose some children (heuristics to figure out #)
  - Record results, use for future play
  - Self-play

- AlphaGo and other big results!




The agent (Black) learns to capture walls and corners in the early game

The agent (Black) learns to force passes in the late game

Credit: Surag Nair

# Summary

- Review of game theory
  - Properties, Mathematical formulation for simultaneous games Normal form, dominance, equilibria, mixed vs pure
- Sequential games
  - Game trees, minimax value, minimax algorithm
- Improving our search
  - Using heuristics, pruning, random search

**Acknowledgements**: Developed from materials by Yingyu Liang (University of Wisconsin), James Skrentny (University of Wisconsin),  inspired by Haifeng Xu (UVA) and Dana Nau (University of Maryland).