CS 577: Introduction to Algorithms — Homework 8

Out: 11/17/10 — **Due: 11/24/10**

Ground Rules

- **Grading.** You will be graded on the correctness as well as clarity of your solutions. You are required to prove any claims that you make. In particular, when you are asked to design an algorithm, you must argue its correctness as well as running time. You may use without proof any theorems proven in class or in the textbook, as long as you state and cite them properly.
- **Collaboration.** You may work on and submit solutions in pairs.
- **Lateness.** Homework is due promptly at the start of class. Late homework will receive zero credit.
- **Extra credit questions.** Extra credit questions will not directly contribute to your score. However, they will be taken into account in the final grading and may improve your overall grade if your total score is close to the boundary between two grades. Furthermore they are fun to solve and improve your understanding of the course.
- Start working on your homework early. Plan your work in such a way that you have the opportunity to put some problems on the back burner for a while and revisit them later. Good luck!

Problems

1. **(5 points.)** Prove that the number of distinct global min-cuts in an undirected graph with n vertices is at most $n(n-1)/2$. Give an example where the number of distinct min-cuts is exactly $n(n-1)/2$. (Note that there are $2^{n-1} - 1$ distinct cuts in all.)

2. **(5 points.)** Problem 13.12 in the textbook (Pg. 790).

3. **(5 points.)** Problem 13.9 in the textbook (Pg. 788–789).

   Hint: First, you don't want to accept the first few bids, because you don't get a chance to compare them to other bids, and the highest one is unlikely to be among the first few. Second, you should use the information in the first few bids to later reject bids that you know are not the highest bid (e.g. those that are lower than some bid you rejected earlier). Note that it is possible to get a success probability better than $1/4$ but you only need to prove a $1/4$ for this homework.

4. **(Extra credit.)** Consider the following game. Alice tosses a fair coin until it comes up heads. She takes two envelopes; If the first head is seen on the kth toss (an event that happens with probability $1/2^k$), she puts $x = 3^k$ dollars in one envelope and $3x$ dollars in the other. She seals the envelopes and gives one of them to Bob. Bob opens the envelope to see the amount in it. (Bob does not see the coin flips, and so does not know the amounts in the envelopes beforehand.) Bob can either keep this money, or ask for the other envelope and keep that amount.

   Now consider Bob's strategy. Suppose that Bob opens the envelope and it contains y dollars. The other envelope can contain either $3y$ or $3/y$ dollars. The first possibility has half the probability of occurence compared to the second possibility. So, in expectation, the amount of money in the second envelope is $1/3(3y) + 2/3(y/3) = 11y/9 > y$. So Bob is better off in expectation asking for the second envelope. However, this decision is independent of what y was!We have an apparent paradox: regardless of which envelope Alice shows to Bob, he is better off asking for the other one.

   Explain this paradox.