CS 577: Introduction to Algorithms	Homework 3
Out: 02/21/12	**Due: 02/28/12**

Ground Rules

Same as for homework 1.

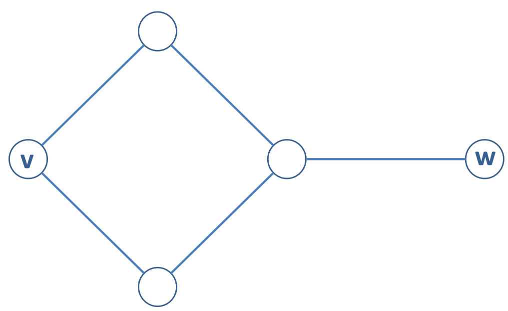
Problems

1. **(4 points)** Problem 3.6 in the textbook (Pg. 108).

   Hint: In both BFS and DFS, the "back edges" are those that do not belong to the corresponding tree. Prove that in DFS back edges always connect a node to its ancestor (that is, another node on the tree path from this node to the root). Prove that in BFS back edges always connect nodes in the same level, or one level apart. Now think of the consequence of the same tree being both the BFS and the DFS tree.

2. **(4 points.)** Often there are multiple shortest paths between two nodes of a graph. Give a linear-time algorithm for the following task: given an undirected graph $G = (V, E)$ with unit edge lengths and nodes v and w, output the number of distinct shortest paths from v to w. For example, for the graph below, on input v and w your algorithm should output 2. (See problem 3.10 in the book for the "story" behind this question.)

   Hint: Think about modifying BFS.



3. **(3+4 points.)** In class we saw that to prove the correctness of Dijkstra's algorithm we need to assert that all edges weights in the graph are nonnegative. In this problem we examine this assumption closely.

   (a) Consider the following algorithm for finding shortest paths in graphs with negative edge weights: add a large constant to each edge weight so that all the weights become positive; then run Dijkstra's algorithm. Is this algorithm correct? Prove your answer.

   (b) Suppose that all of the negative weight edges are those that leave the source node s. Can Dijkstra's algorithm, started at s, fail on such a graph? Prove your answer.

4. **(Extra credit.)** Consider the following game. Alice tosses a fair coin until it comes up heads. She takes two envelopes; If the first head is seen on the kth toss (an event that happens with probability $1/2^k$), she puts $x = 3^k$ dollars in one envelope and $3x$ dollars in the other. She seals the envelopes and gives one of them to Bob. Bob opens the envelope to see the amount in it. (Bob does not see the coin flips, and so does not know the amounts in the envelopes beforehand.) Bob can either keep this money, or ask for the other envelope and keep that amount.

   Now consider Bob's strategy. Suppose that Bob opens the envelope and it contains y dollars. The other envelope can contain either $3y$ or $3/y$ dollars. The first possibility has half the probability of occurence compared to the second possibility. So, conditioned on y, in expectation the amount of money in the second envelope is $1/3(3y)+2/3(y/3) = 11y/9 > y$. So Bob is better off in expectation asking for the second envelope. However, this decision is independent of what y was! We have an apparent paradox: regardless of which envelope Alice shows to Bob, he is better off asking for the other one.

Explain this apparent paradox.