CS880: Approximation and Online Algorithms	**Scribe:** Kexin Li
Lecture 2: Sparsest Cut	**Date:** 2019/09/16

2.1 LP Relaxation of Sparsest cut

Definition 2.1.1 (Sparsest Cut Problem) *Given $G = (V, E)$ and $c : E \to \mathbb{R}^+$, find a partition of V into $(S, V \backslash S)$ where $|S| \leq \frac{n}{2}$ to minimize the sparsity $\phi(S) = \frac{c(\delta(S))}{|S||V \backslash S|}$*

In general, the objective function is $\phi(S) = \frac{c(\delta(S))}{demand(S)}$. The demand of sparsest cut problem is number of pairs cross the cut, which equals to $|S||V \backslash S|$. The sparsest cut is going to separate the graph to 2 large components in the order of n. In this case, it will look like small balance cut with nice algorithmic application to be used in divide and conquer style. In a different context, we could sample the vertices with certain probabilities by some random walk on a graph. The total cost of edges leaving set S relative to its size gives an indication of likelihood of a random walk to step outside the set. Observe that the sparsity of a graph is symmetric with respect to the set S and its complement. So it's usually defined for $|S| \leq \frac{n}{2}$. Therefore $\phi(S)$ is between 1 or 2 times of $\frac{c(\delta(S))}{|S|}$

The ILP of cut formulation is

$$\begin{aligned} &\text{minimize } \frac{\sum_e c_e d_e}{\sum_{u,v} d_{uv}} \\ &\text{subject to } d \text{ is a cut metric} \\ &d \in \{0, 1\} \end{aligned} \tag{2.1.1}$$

Consider all the edges leaving the component S_i, they are counted twice if we sum over all the components. The same applies to the demands.

$$c(S_1, S_2, \ldots, S_k) = 2 \sum_i c(\delta(S_i))$$

$$dem(S_1, S_2, \ldots, S_k) = \sum_{u \in S_i, v \in S_j, i \neq j} \mathbb{1} = 2 \sum_i dem(S_i, V \backslash S_i)$$

The sparsity of the entire partition is at least as good as the best individual cut. It never helps to partition the graph to more than two components. Another implication is the sparsest cut is easy over trees by removing one edge.

$$\frac{c(S_1, S_2, \ldots, S_k)}{dem(S_1, S_2, \ldots, S_k)} = \frac{\sum_i c(\delta(S_i))}{\sum_i dem(S_i, V \backslash S_i)} \geq \min_i \frac{c(\delta(S_i))}{dem(S_i, V \backslash S_i)}$$

Linear combination of metrics is also a metric because triangle inequality holds over linear combination. The convex hull of a collection of cut metric is all cut metric, which we define as *cut cone*.

Definition 2.1.2 *Cut cone is all metrics obtained as positive linear combination of cut metrics*

We can relax the constrains of d is a cut metric with $d \in$ cut cone. And based on previous observation, the sparsity of d is not better than one of the cut metrics in the linear combination of d. If we could optimize the objective over the cut cone, then we will get one cut metric from it. This is not necessarily true for other cut problems. For example, it's not true for max cut problem and multicut problem because we may have linear combination of cut metrics to separate all terminal pairs but no single cut metric can.

Claim 2.1.3 *The cut cone is equivalent to the set of all l_1 metrics.*

Proof: Let l_1 metric is a map from vertex u, v to vector X^u, X^v in $\mathbb{R}^n$ and $|X^u - X^v|_1 = \sum_{i=1}^n |X_i^u - X_i^v|$. Because cut metric is l_1-metric and l_1-metric is closed under linear combination by appending and rescaling in each dimension, cut cone is in l_1-metric. Let's consider the other direction -l_1-metric is in cut cone. We embed distance from l_1 to cut cone in each dimension and sum them over n. We first project all points to one single dimension and place a cut for each adjacent pairs of points. Then the linear combination of those cut metrics is equivalent to l_1-metric in that dimension.

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Therefore we can optimized (2.1.1) over d is l_1-metric without considering the scaling. This is NP-hard and it captures the exact sparsest cut problem.

$$\begin{aligned} \text{minimize} & \sum_e c_e d_e \\ \text{subject to } & d \in l_1 \text{ metric} \\ & \sum_{u,v} d_{uv} = 1 \end{aligned} \tag{2.1.2}$$

We can relax the constrain to general metric to solve it efficient. If general metric embed into l_1 with distortion α, then it's the α approximation to sparsest cut. Recall this relaxation from cut metric to general metric is exactly what we did for multiway cut and multicut.

$$\begin{aligned} \text{minimize} & \sum_e c_e d_e \\ \text{subject to } & d \in l_1 \text{ metric} \\ & \sum_{u,v} d_{uv} = 1 \end{aligned} \tag{LP relaxition}$$

Theorem 2.1.4 (Borgains Theorem) *Every n points metric embeds into l_1 metric with distortion $\alpha = O(\log n)$ in $O(\log^2 n)$ dimension*

Tree metric is also l_1 metric. Because a tree can be represented by linear combination of cut metric where each cut metric is represented by one edge with weight of edge length. Any graph can have exponential many trees in it. Leighton-Rao first discovered this method of LP relaxation to general metric with $O(\log n)$ approximation factor. The integral gap of this LP is $\Omega(\log n)$ so we cannot have better approximation with this LP.

2.2 SDP Relaxation of Sparsest Cut

Let's think of another way to formulate sparsest cut problem with vector dot product. We map vectors in S to -1 and $V \backslash S$ to 1. Later we can obtain a SDP relaxation of this problem.
First, this is the exact description of sparsest cut problem.

$$\begin{aligned} &\text{minimize } \frac{\frac{1}{2}\sum_e c_e(1 - v_i v_j)}{\frac{1}{2}\sum_{i,j}(1 - v_i v_j)} \\ &\text{subject to } v_i \cdot v_i = 1 \end{aligned} \tag{2.2.3}$$

We can linearize the ratio by scaling so the denominator equals to 1.

$$\begin{aligned} \text{minimize } &\sum_e c_e(1 - v_i v_j) \\ \text{subject to } &v_i \cdot v_i = v_j \cdot v_j, \forall i, j \\ &\sum_{i,j}(1 - v_i v_j) = 1 \end{aligned} \tag{2.2.4}$$

Next since all the vectors lies on some sphere, we can rescale them to be on the unit sphere.

$$\begin{aligned} \text{minimize } &\sum_e c_e(1 - v_i v_j) \\ \text{subject to } &v_i \cdot v_i = 1, \forall i \\ &\sum_{i,j}(1 - v_i v_j) = S^* \end{aligned} \tag{2.2.5}$$

The square length of difference of two vectors $|v_i - v_j|^2 = v_i \cdot v_i + v_j \cdot v_j - 2v_i \cdot v_j = 2(1 - v_i v_j)^2$ plays the role of d_{ij}. Then we can rewrite the sum

$$\begin{aligned} \text{minimize } &\frac{1}{2} \sum_{(i,j)\in E} c_{i,j}|v_i - v_j|^2 \\ \text{subject to } &v_i \cdot v_i = 1, \forall i \\ &\frac{1}{2}\sum_{i,j} |v_i - v_j|^2 = S^* \end{aligned} \tag{2.2.6}$$

Recall Geomans-Williamson method from last time. We want the contribution from random cut is not much large than the contribution from SDP. However, the rescale of the linear line is unbounded.

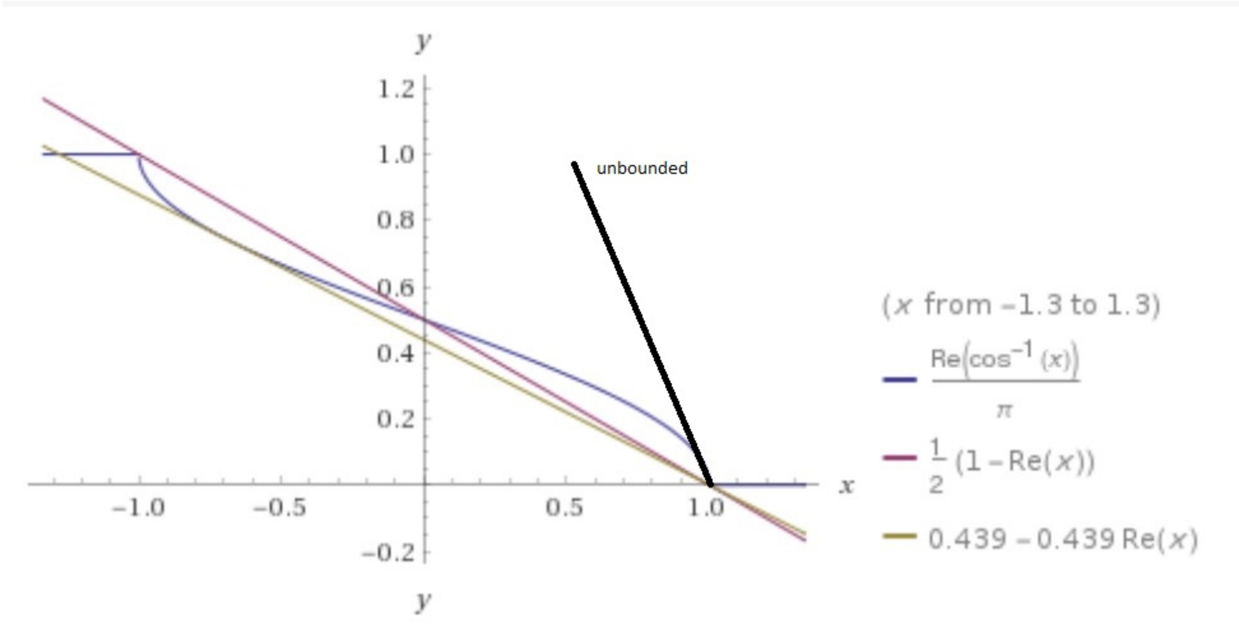

Figure 2.2.1: Approximation Factor of Geomans-Williamson Randomazition SDP method

Because the square of Euclidean length may not be metric, we want to impose both constrains.

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} c_{ij} d_{ij} \\
\text{subject to} \quad & \sum_{i,j} d_{ij} = S^* \\
& \{d_{ij}\} \text{ is a metric} \\
& \{\sqrt{d_{ij}}\} \text{ is an Euclidean metric}
\end{aligned} \tag{Goemans-Linial Relaxation}$$

Definition 2.2.1 *Square Euclidean metric l_2^2 is called negative-type metric and it satisfied $\{d_{ij}\}$ is a metric and $\{\sqrt{d_{ij}}\}$ is an Euclidean metric*

The integral gap for this formulation is $\Omega(\log\log n)$ and $O(\sqrt{\log n})$. We will talk about this result in next lecture. Arora Rao Vazirani showed that $l_2^2 \hookrightarrow l_1$ is a contracting embedding where the average distortion is $O(\sqrt{\log n})$. And in a followup by Lee Naor, they showed the distortion is $O(\sqrt{\log n}\log\log n)$

References

[1] Goemans, Michel X and Williamson, David P Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. In *Journal of the ACM (JACM)*, 1995.

[2] Arora, Sanjeev and Rao, Satish and Vazirani, Umesh Expander flows, geometric embeddings and graph partitioning. In *Journal of the ACM (JACM)*, 2009.

[3] Arora, Sanjeev and Lee, James and Naor, Assaf Euclidean distortion and the sparsest cut. In *Journal of the American Mathematical Society*, 2008.