CS880: Approximation and Online Algorithms	**Scribe:** Xiating Ouyang
Lecture 8: Iterative rounding continued and Primal Dual Algorithms	**Date:** 10/2/2019

8.1 Degree-bounded Spanning Tree Problem

In this scribe we revisit the Degree-bounded Spanning Tree Problem: Given a graph $G = (V, E)$ with cost on each edge e being c_e, and degree bounds B_v for each $v \in W$, find a minimum cost spanning tree over G such that $deg_T(v) \leq B_v$ for each $v \in W$.

We denote $\delta(v)$ as the set of edges in G incident to v and $E(S)$ be the set of edges whose both endpoints are in S. We introduce a linear program modelling this problem: Introduce a variable x_e for each edge e, with $x_e = 1$ denoting x_e is chosen in the spanning tree, and $x_e = 0$ otherwise. Consider the following linear program:

$$
\begin{aligned}
\text{min} \quad & \sum_{e \in E} c_e x_e & & \\
\text{subject to} \quad & \sum_{e \in E(S)} x_1 \leq |S| - 1 & \forall S \subset V, |S| \geq 2 \quad & (1) \\
& \sum_{e \in E} x_e = |V| - 1 & & (2) \\
& \sum_{e \in \delta(v)} x_e \leq B_v & \forall v \in W \quad & (3) \\
& x_e \geq 0 & \forall e \in E \quad & (4)
\end{aligned}
$$

Constraint (1) enforces connectivity (each vertex subset does not contain sufficient edges to form a cycle); (2) enforces that the resulting edges forms a tree; and (3) enforces the degree bound. Note that though this LP contains exponential number of constraints, it can be solved efficiently using the ellipsoid method due to Cunningham [1984] and Lau et al. [2011].

Let us consider the polytope P enclosed by the constraints (1)–(4). The following lemma results in an iterative rounding algorithm for our problem:

Lemma 8.1.1 *For any extremal point x of P, exactly one of the following holds:*

(i) there exists some edge e with $x_e = 0$;

(ii) there exists some edge e with $x_e = 1$; and

(iii) there exists some vertex $v \in W$ with $deg_E(v) \leq 3$.

Our algorithm proceeds as follows: Solve the LP and yield an optimal solution x. For each edge e with $x_e = 0$, remove e from the graph; for each edge e with $x_e = 1$, include it in the solution; and for each vertex v with $dev_E(v) \leq 3$, we drop v from W, and then resolve the new LP iteratively.

The algorithm will terminate in at most $|E| + |W|$ steps since in each step we either remove one variable or one constraint from the LP. Moreover, the algorithm never makes the LP result worse since our modification to the LP respects the value of each x_e in the optimal solution. Since we remove the vertex v with $v \in W$ with $deg_E(v) \leq 3$ and in the worst case, $B_v = 1$, the spanning tree returned by the algorithm violates the degree bound by at most 2.

Now it remains to show Lemma 8.1.1 is true, for which we need the following lemma.

Lemma 8.1.2 *For any extremal point $x \in P$, there exists a family of sets $\mathcal{L} \in 2^V$ and a subset of vertices $Z \in W$ such that*

(i) the constraints corresponding to $S \in \mathcal{L}$ and vertices $v \in Z$ are tight;

(ii) the constraints corresponding to L and Z are linearly independent;

(iii) $|\mathcal{L}| + |Z| = |E|$; and

(iv) $\mathcal{L}$ is a laminar family.

Let $\mathcal{L}$ be a family of sets of 2^V. The family $\mathcal{L}$ is a *laminar family* if for any distinct $A, B \in \mathcal{L}$, either $A \cap B = \emptyset$, or $A \subseteq B$ or $B \subseteq A$. If each set in $\mathcal{L}$ contains at least 2 elements (due to (1) restricting $|S| \geq 2$), we have that $|\mathcal{L}| \leq |V| - 1$.

In what follows we show the proof of Lemma 8.1.1 and Lemma 8.1.2.

Proof of Lemma 8.1.1

First, if there exists some vertex v such that $deg_E(v) = 1$, let the edge incident to v be e, and we must have $x_e = 1$, and thus (ii) holds. We thus consider the case where $deg_E(v) \geq 2$ for each $v \in V$.

Suppose that (i) and (ii) do not hold in Lemma 8.1.1, and for contradiction that (iii) does not hold. In this case, we have $x_e > 0$ for each edge e, and that for each $v \in W$, $deg_E(v) \geq 4$. Hence we have

$$\begin{aligned} |E| = \frac{1}{2}\sum_{v \in V} deg_E(v) &= \frac{1}{2}(\sum_{v \in W} deg_E(v) + \sum_{v \notin W} deg_E(v)) \\ &\geq \frac{1}{2}(\sum_{v \in W} 4 + \sum_{v \notin W} 2) \\ &= \frac{1}{2}(4|W| + 2(|V| - |W|)) \\ &= |V| + |W|. \end{aligned}$$

However by Lemma 8.1.2, we have that

$$|E| = |\mathcal{L}| + |Z| \leq |V| - 1 + |W|,$$

a contradiction. Thus (iii) holds.

Proof of Lemma 8.1.2

Since the constraints of the LP are linear and the variable x is in an $|E|$ dimensional space, any extremal point is the intersection of precisely $|E|$ linearly independent tight constraints. In our setting, there are $|\mathcal{L}|$ constraints in constraints (1) and (2) and $|Z|$ constraints in constraints (3), and thus (i), (ii) and (iii) follow.

To show (iv), it suffices to show the following lemma.

Lemma 8.1.3 *Given any family $\mathcal{F} \subseteq 2^V$ of the tight sets in the LP, there exists a laminar family $\mathcal{L} \subseteq 2^V$ of linearly independent tight sets such that span$(\mathcal{F}) \subseteq$ span$(\mathcal{L})$.*

Proof: Let $\mathcal{L}$ be a maximal laminar family of linearly independent tight sets of $\mathcal{F}$. Here $\mathcal{L}$ is maximal if the family $\mathcal{L} \cup \{S\}$ is not a laminar family, though S is tight and $\mathcal{L} \cup \{S\}$ are linearly independent.

Suppose for contradiction that there exists some tight set in *span*$(\mathcal{F})$ that is not in $\mathcal{L}$, and let S be the one set that intersects as few sets in $\mathcal{L}$ as possible. Consider any set $T \in \mathcal{L}$. We denote $x(S) := \sum_{e \in E(S)} x_e$ and show the following three claims.

Claim 8.1.4 *Sets $S \cap T$ and $S \cup T$ are tight.*

Since S and T are tight, we have

$$x(S) = |S| - 1 \text{ and } x(T) = |T| - 1.$$

We also have

$$x(S \cup T) \leq |S \cup T| - 1 \text{ and } x(S \cap T) \leq |S \cap T| - 1.$$

It then follows that

$$x(S \cap T) + x(S \cup T) \leq |S \cap T| + |S \cup T| - 2 = |S| + |T| - 2 = x(S) + x(T).$$

On the other hand, any edge appearing in S or T must appear in $S \cup T$ or $S \cap T$, and $S \cup T$ may contain edges whose endpoints are in $S \setminus T$ and $T \setminus S$ respectively. This yields

$$x(S \cap T) + x(S \cup T) \geq x(S) + x(T),$$

which implies that both $S \cap T$ and $S \cup T$ are tight.

Claim 8.1.5 *Let $\chi(F)$ be the membership vector of each edge $e \in F$. Then $\chi(S \cap T) + \chi(S \cup T) = \chi(S) + \chi(T)$.*

Claim 8.1.4 shows that there exists no edge whose endpoints are in $S \setminus T$ and $T \setminus S$ respectively, and thus each edge is counted in S or T on the right hand side is counted in the left hand side and vice versa.

Claim 8.1.6 *The sets $S \cup T$ and $S \cap T$ intersect with fewer sets in $\mathcal{L}$ than S.*

This follows since any set intersecting $S \cap T$ is intersecting S, and $S \cap T$ does not intersect with T, and S does otherwise.

Since $S \notin$ *span*$(\mathcal{L})$, by Claim 8.1.5 we have either $S \cup T \notin$ *span*$(\mathcal{L})$ or $S \cap T \notin$ *span*$(\mathcal{L})$. By Claim 8.1.4 and 8.1.6, both $S \cap T$ and $S \cup T$ are tight and intersects with fewer sets in $\mathcal{L}$, contradicting that S is the set intersecting with the fewest sets in $\mathcal{L}$. Thus *span*$(\mathcal{F}) \subseteq$ *span*$(\mathcal{L})$. ■

8.2 Introduction to LP Duality

Consider the following linear program P_1:

$$\begin{array}{ll} \min & 5x + 3y \\ \text{subject to} & x + y \geq 10 \\ & x + 2y \geq 15 \\ & x \geq 0 \\ & y \geq 0. \end{array}$$

If we sum up the first two constraints, we would yield $(x+y)+(x+2y) = 2x+3y \geq 10+15 = 25$. Since $5x + 3y \geq 2x + 3y$, we have obtained 25 as a lower bound of $5x + 3y$. More generally, if we multiply the first and second constraint by nonnegative multipliers α_1 and α_2 and sum them up, we would have

$$\alpha_1(x + y) + \alpha_2(x + 2y) \geq 10\alpha_1 + 15\alpha_2,$$

that is

$$(\alpha_1 + \alpha_2)x + (\alpha_1 + 2\alpha_2)y \geq 10\alpha_1 + 15\alpha_2.$$

Then for any nonnegative α_1 and α_2 satisfying

$$\alpha_1 + \alpha_2 \leq 5 \text{ and } \alpha_1 + 2\alpha_2 \leq 3,$$

the value $10\alpha_1 + 15\alpha_2$ is a lower bound of $5x + 3y$, and thus is the largest that $10\alpha_1 + 15\alpha_2$ can achieve. That is, if we consider the program P_2:

$$\begin{array}{ll} \max & 10\alpha_1 + 15\alpha_2 \\ \text{subject to} & \alpha_1 + \alpha_2 \leq 5 \\ & \alpha_1 + 2\alpha_2 \leq 3 \\ & \alpha_1 \geq 0 \\ & \alpha_2 \geq 0, \end{array}$$

any feasible solution of P_2 is a lower bound on the optimal solution of P_1, called the *weak duality theorem*. The LP P_2 is called the dual of P_1.

In general, consider the linear program P, then the linear program D is the *dual* of P.

(P)

$$\begin{array}{ll} \min & c^\top x \\ \text{subject to} & Ax \geq b \\ & x \geq 0. \end{array}$$

(D)

$$\begin{array}{ll} \min & b^\top y \\ \text{subject to} & A^\top y \leq c \\ & y \geq 0. \end{array}$$

Theorem 8.2.1 (Weak Duality Theorem) *Let x and y be feasible solutions of P and D respectively. Then $c^\top x \geq b^\top y$.*

Proof: By feasibility of x and y, it follows that

$$c^\top x \geq (A^\top y)^\top x = y^\top Ax = y^\top (Ax) \geq y^\top b = b^\top y.$$

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The strong duality theorem states that if x^* and y^* are the optimal solutions of P and D respectively, then we have $c^\mathsf{T} x^* = b^\mathsf{T} y^*$. Moreover, the following conditions hold for x^* and y^*, called the *complementary slackness*:

- For each i, we have either $x_i^* = 0$ or $(A^\mathsf{T} y^*)_i = 0$; and
- For each j, we have either $y_j^* = 0$ or $(Ax^*)_j = 0$.