Today:

Multi-Parameter Mechanism Design.

HW1 due today

HW2 out later this week. — due Nov 2

<u>Projects</u>

- Report — due Dec 14.
- 1hr presentation. — between Nov 23 & Dec 14.
- Oct 22/23 — half hr. meetings —

- Ideas for topics on webpage.

Some observations about single parameter MD.

- Characterization of BIC mechs. — monotonicity
  — payment identity.
- Revenue optimal mechs as virtual value maximizers.

Some multi-parameter valuations. (combinatorial valuations)

① Unit-demand.

many different kind of items/services — indexed by i.
$i \in [m]$

Buyer's value — vector $v_1, v_2, \dots, v_m$.
v_i — value assigned to item i.

$$V(S) = \max_{i \in S} v_i .$$

↑ value ↖ set of items.

x — allocation vector
x_i = Pr that item i is alloc.
$\sum_i x_i = 1$
$V(x) = x \cdot v$

② Additive $V(S) = \sum_{i \in S} v_i$

Problem: 1 buyer, 1 seller with m items. $v \sim F$

F → joint distribution over m-dimensional vectors.

What is the optimal selling mechanism?.

Solution in single parameter case: monopoly reserve price set on the item.

TAXATION PRINCIPLE: Same allocations should always be charged the same price.

Implication: Describe any mechanism as a menu of options where each option is an (allocation, price) pair, and allow the buyer to choose their fav. option.

E.g. $v = (1,2)$

$x = (1,0)$

$\text{Value}(x) = 1\cdot 1 + 2\cdot 0 = 1$

$x' = (1/2, 1/2)$

$\text{Value}(x') = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = 1\frac{1}{2}$

E.g. 2 items. 3 possible types of buyers — A: $(1,0)$; B: $(0,1)$; C: $(\frac{1}{2},\frac{1}{2})$

F — uniform over these types

Optimal Deterministic Menu = $\left\{ \begin{array}{ll} x, & p \\ (1,0), & 1 \\ (0,1), & 1/2 \end{array} \right\}$

- $(1,0), 1$ — A
- $(0,1), 1/2$ — B, C

Revenue: $1\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{2}{3} = \frac{2}{3}$

$\left\{ \begin{array}{ll} (1,0), & 1 \\ (0,1), & 1 \end{array} \right\}$

- $(1,0), 1$ — A
- $(0,1), 1$ — B
- X — C

Revenue: $1\cdot\frac{2}{3} + 0\cdot\frac{1}{3} = \frac{2}{3}$

Given: item pricing (p_1, p_2), a buyer with value (v_1, v_2) will select $\arg\max_i (v_i - p_i)$.

Menu with lottery tickets = $\left\{ \begin{array}{ll} (1,0) & 1-\varepsilon \\ (0,1) & 1-\varepsilon \\ (\frac{1}{2},\frac{1}{2}) & \frac{1}{2} \end{array} \right\}$

- $(1,0), 1-\varepsilon$ — A
- $(0,1), 1-\varepsilon$ — B
- $(\frac{1}{2},\frac{1}{2}), \frac{1}{2}$ — C (lottery ticket)

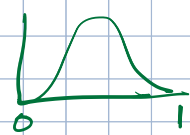
Revenue: $1\times\frac{1}{3} + 1\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3} = \frac{5}{6}$

Even with unit-demand buyers over 2 items,
the optimal mechanism may have a menu of unbounded size.

Example from Daskalakis, Deckelbaum, Tzamos.

Additive buyer over 2 items.

$v_1 \sim \text{Beta}(3,3)$



$v_2 \sim \text{Beta}(3,4)$

Optimal menu has unbounded size.

In fact, no approx. for optimal revenue via bounded size menus.

Duality Based Approach for characterizing revenue optimal mechanisms.

One buyer. T — set of possible types. $t \in T$.

x_t : allocation at type t $\qquad t \cdot x_t$ = value from alloc.

p_t : payment at type t.

f_t : prob. mass over type t.

$$\max_{(x,p)} \sum_t f_t \cdot p_t$$

s.t. x_t is "supply-feasible" $\Leftrightarrow x \in P \quad \forall t \in T$

$\lambda_{t,t'}$ IC $\quad t \cdot x_t - p_t \geq t \cdot x_{t'} - p_{t'} \quad \forall t, t' \in T$

$x \geq 0.$

$$\max_{\substack{(x,p)\\ x \in P}} \underbrace{\sum_t f_t \cdot p_t + \sum_{t,t'} \lambda_{tt'}\left(t\cdot(x_t - x_{t'}) - (p_t - p_{t'})\right)}_{\mathcal{L}(\lambda, x, p)}$$

Observe: For any choice of $\lambda \geq 0$ and any BIC (x,p), $\mathcal{L}(\lambda, x, p) \geq Rev(x,p)$

Corollary: For any $\lambda \geq 0$. $\max_{\substack{(x,p)\\ x \in P}} \mathcal{L}(\lambda, x, p)$ is an upper bound on OPT.

New problem:

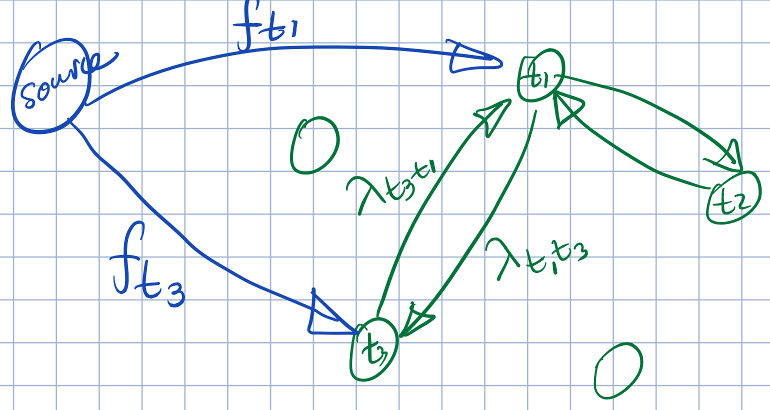
$$\min_{\lambda \geq 0} \max_{\substack{(x,p)\\ x \in P}} \mathcal{L}(\lambda, x, p).$$

Definition: λ is useful if $\max_{\substack{x,p\\ x \in P}} \mathcal{L}(\lambda, x, p) < \infty.$

$$\mathcal{L}(\lambda, x, p) = \sum_{t,t'} \lambda_{tt'}\, t\cdot(x_t - x_{t'}) + \sum_t p_t \left\{ f_t - \sum_{t'} \lambda_{tt'} + \sum_{t'} \lambda_{t't} \right\}$$

In order for $\max_{\substack{x,p\\ x \in P}} \mathcal{L}(\lambda, x, p) < \infty$, it must be the case that $\forall t$,

$$\boxed{f_t + \sum_{t'} \lambda_{t't} = \sum_{t'} \lambda_{tt'}}.$$ flow conservation



For any useful $\lambda \geq 0$, we can rewrite

$$\mathcal{L}(\lambda, x, p) = \sum_t x_t \left\{ \sum_{t'} t \lambda_{tt'} - \sum_{t'} t' \lambda_{t't} \right\}$$

$$\sum_t f_t x_t \cdot \phi_t$$

$$= \sum_t x_t \left\{ t \sum_{t'} \lambda_{tt'} - \sum_{t'} t' \lambda_{t't} \right\}$$

$$= \sum_t x_t \left\{ t f_t + t \sum_{t'} \lambda_{t't} - \sum_{t'} t' \lambda_{t't} \right\}.$$

$$= \sum_t x_t f_t \left\{ t - \frac{\sum_{t'} \lambda_{t't}(t'-t)}{f_t} \right\}$$

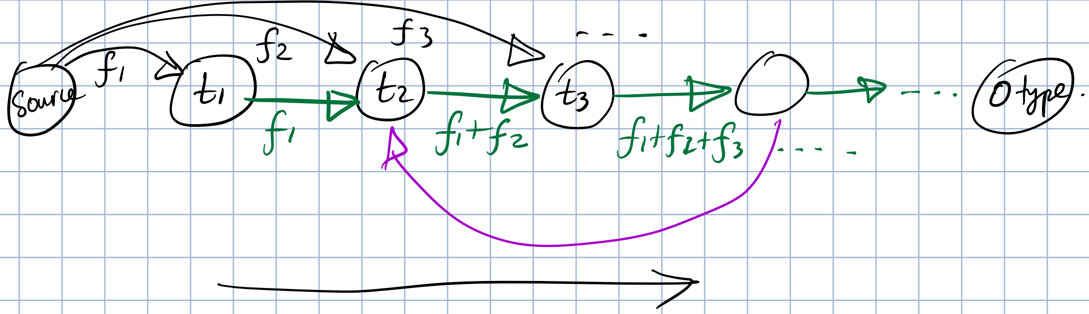
ϕ_t^λ : Virtual value w.r.t. λ at type t.

Myerson's v.v. : $\phi_t = t - \frac{1-F(t)}{f(t)}$

$\mathcal{L}(\lambda, x, p)$: Virtual surplus (x, p).

Lemma: For any useful $\lambda \geq 0$, and any IC (x, p).
Rev (x, p) $\leq$ Virtual surplus.

Special case of single param. settings.



Sort types in decreasing order.

$t_1 \geq t_2 \geq t_3 \geq \dots$

__Unit Demand__ : single buyer. [CHK'07]

Assume : $v \sim F = F_1 \times F_2 \times \dots \times F_m$. Product distribution.

$v = (v_1, v_2, \dots)$ $\quad v_i \sim F_i$.

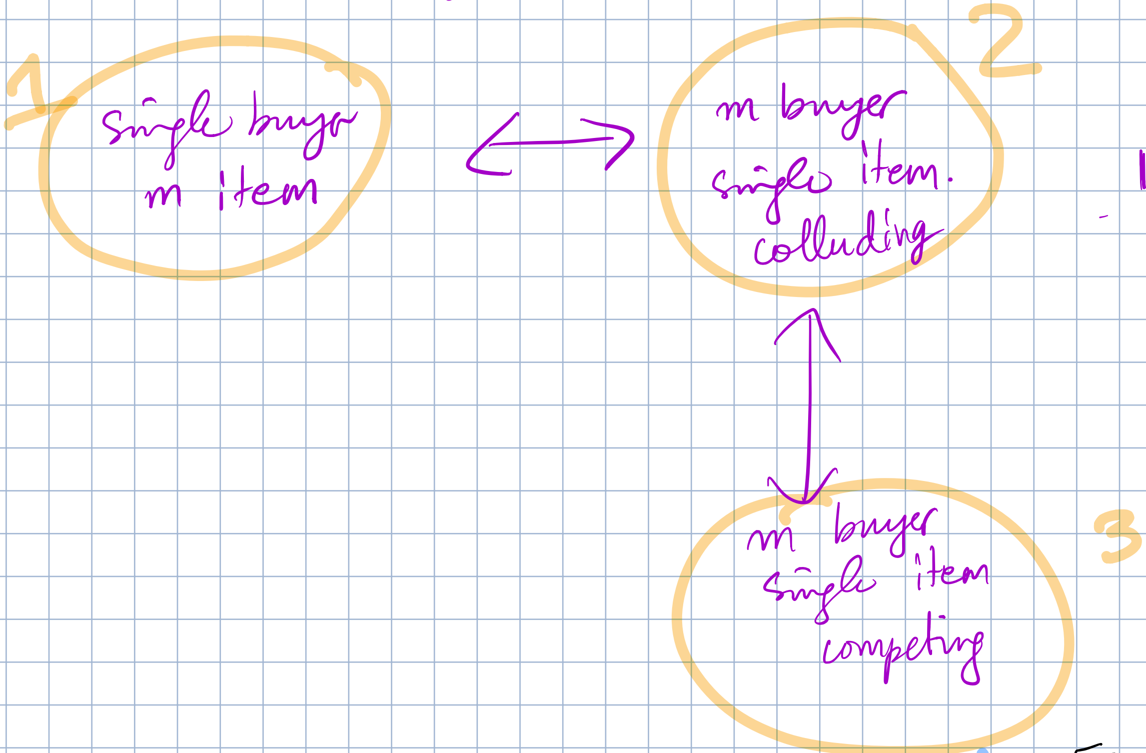
Think of a buyer as m "copies" or colluding friends.

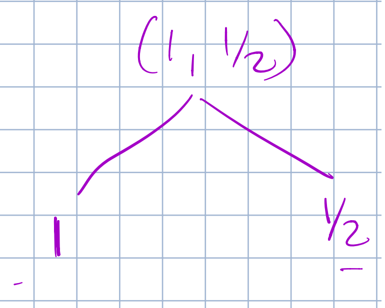
Each "copy" i is only interested in buying item i.

value $- v_i \sim F_i$.

↓

Single parameter.





Does not hold. [Chawla Malec Sivan '10]

Opt rev for 2 $\leq$ Opt rev for 3

$\wedge$|

2. Approx mech for 2 = 2. Approx mech for 3

"collusion - robust"

↓

Order-oblivious Posted Price OPMs.

based on Prophet Inequalities

Item Pricing for 1 ⟸

$\Rightarrow$ In the unit-demand indep. item values setting, above item pricing based on OPMs is a 2-approx. to the optimal revenue.

↓ deterministic

In the unit-demand indep. item setting, item pricing is a 4-approx. to the optimal revenue.