Oct 19

Today: multi-parameter MD approximations.

simple versus optimal

HW 2 out on webpage.

Please sign up for project meetings.

Multi-parameter : many different kinds of outcomes.

multi-dim. value assigned to outcomes.

Combin. auction — $m$ items.

$v(S)$

- unit-demand
- additive
- additive upto $k$.

Assumption in most simple versus optimal results for MDMD.

Values of buyers for different items are independent.

One Unit-Demand buyer with independent values.

— Opt. mech may sell lotteries.

— Item pricing performs approx-optimally.

Opt. → Setting 1: one agent, unit-demand, $v \sim F_1 \times \cdots \times F_m$.

Mye → Setting 2: $m$ agents, single item, $v_i \sim F_i$.

We argued: Item Pricing for setting 1 has revenue $\geq \frac{1}{2}$ Mye.

$\boxed{\text{Claim: } \quad \text{Mye} \geq \frac{1}{2} \text{ Opt.}}$ Holds only for unit dem.

Corollary: Item Pricing $\geq \frac{1}{4}$ Opt. $\frac{1}{10}$.

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Proof of Claim: $(\underline{x}, p)$ $\qquad \sum_i x_i v_i - p$.

At any value vector $v$, fav. item $i^* = \arg\max_i v_i$.

Construct a mech for Setting 2:

When vector $v$ is reported, assign to $i^*$ a fraction $x_{i^*}(v)$ of the items and charge the price

$$p - \sum_{i \neq i^*} x_i(v)\, v_i.$$

Util of friend $i$ from alloc.

$$= x_{i^*}(v) \cdot v_{i^*} - \left(p - \sum_{i \neq i^*} x_i(v)\, v_i\right)$$

$$= \sum_i x_i(v)\, v_i - p.$$

= Util of unit-dem agent from alloc.

$$\text{Rev of new mech} = \mathop{E}_{v}\left[ p - \sum_{i \neq i^*} x_i(v)\, v_i \right]$$

$$= \underbrace{\mathop{E}_{v}[p]} - \mathop{E}_{v}\left[ \sum_{i \neq i^*} x_i(v)\, v_i \right]$$

$$\geq \text{Opt.} - \mathop{E}_{v}\left[ \max_{i \neq i^*} v_i \right]$$

$$\geq \text{Opt} - \text{Rev of Vickrey auction in Setting 2.}$$

$$2\,\text{Mye} \geq \underset{\text{for Setting 2}}{\text{Rev of new mech}} + \underset{\text{for Setting 2}}{\text{Rev of VA}} \geq \text{Opt.}$$

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## Additive Values.

$m$ items, one agent.

$v_i \sim F_i$. $v_i$'s are independent.

$v(S) = \sum_{i \in S} v_i$.

Ex. 1. 2 items $v_1, v_2 \sim \text{Unif}\{1,2\}$.

Price each item separately, rev = 1 per item or \$2 in all.

Sell the two items as a bundle

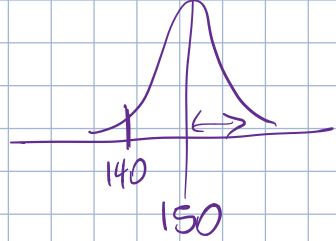
dist of sum of values – $\text{Unif}\{2,3,3,4\}$

opt. bundling rev $= 3 \cdot \frac{3}{4} = \frac{9}{4}$

Ex. 2. 100 items with $v_i \sim \text{Unif}\{1,2\}$.

Sell Separately Rev $\text{SRev} = 100$

Bundling Rev $\text{BRev} \approx 150 - \text{something small}$.




$\downarrow$ concentration of sums!.

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Ex. 3. $n$ items. $v_i \sim \begin{cases} 0 & 1-\frac{1}{n2^i} \\ 2^i & \frac{1}{n2^i} \end{cases}$

$\sum_i v_i$ — Equal Rev variable (discrete version)

$F(x) = 1-\frac{1}{x}$ $[1,\infty)$

$Rev(p) = p(1-F(p)) = p \cdot \frac{1}{p} = 1$

SRev $\gg$ BRev.

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Ex. 4. 2 items. $v_1 \sim Unif\{1,2\}$.
$v_2 \sim Unif.\{1,3\}$.

$v_1+v_2 \sim Unif\begin{cases} 2 & A \\ 3 & B \\ 4 & C \\ 5 & D \end{cases}$

SRev $= 1.1 + 3.\frac{1}{2} = 2.5$.

BRev $= 3.\frac{3}{4} = \frac{9}{4} = 2.25$.

Opt. menu $= \begin{cases} (1,1), & \$4 \\ (1,\frac{1}{2}), & \$2.5 \\ (0,0), & \$0. \end{cases}$ D, C, B, A

Opt $= 4.\frac{1}{2} + 2.5\frac{1}{4} = 2.625$.

Any mech. is a menu of options — (alloc, price)

random subset or dist over subsets.

$(\frac{1}{2},\frac{1}{2}) \Longleftrightarrow \frac{1}{2}\{1,2\} + \frac{1}{2}\phi$ or $\frac{1}{2}\{1\} + \frac{1}{2}\{2\}$.

Main Result. [Babaioff, Immorlica, Lucier, Weinberg '14].

$$\mathrm{Max}(\mathrm{SRev}, \mathrm{BRev}) \geq \frac{1}{6} \mathrm{Opt}.$$

Proof

Rev $\longleftrightarrow$ SRev

Lemma 1: Let $S$ and $T$ be disjoint sets of items.
let $F$ be a dist. over values over $S$.
and $F'$ —"— item values over $T$.

$$\boxed{\mathrm{Rev}(F \times F') \leq \underbrace{\mathrm{Rev}(F)}_{\text{"Tail"}} + \underbrace{\mathrm{Val}(F')}_{\text{"Core"}}}$$

$$\mathrm{Val}(F') = \underbrace{\mathop{E}_{F'}\left[\sum_{i \in T} v_i\right]}_{\text{Want this to be concentrated.}}$$

Proof: Start with opt. mech $M$ for $F \times F'$.
Define a mech. $M'$ for $F$.
Given $v_{|S} \sim F$.
Draw $v_{|T} \sim F'$. Gives us a draw from $F \times F'$.

$M$ assigns $(x, p)$ at $v$.

$M'$ will assign $x_{|S}$ to $v_{|S}$.
charge $p - \sum_{i \in T} x_i v_i$.

Claim: $M'$ is BIC.

$$Rev(M') = Rev(M) - \mathop{E}_{v \sim F'}\left[\sum_{i \in T} x_i v_i\right]$$

$$\leq Rev(M) - \underbrace{\mathop{E}_{v \sim F'}\left[\sum_{i \in T} v_i\right]}_{Val(F')}.$$

## Core-Tail Decomposition

For a random $v \sim \underbrace{F_1 \times \cdots \times F_m}_{F}$,

$i$ is in Core if $v_i \leq t_i$

$i$ is in Tail if $v_i > t_i$

$q_i = Pr[i \in Tail] = Pr[v_i > t_i]$

Note: $t_i \underbrace{Pr[v_i > t_i]}_{q_i} \leq r_i$

$q_i \leq \dfrac{r_i}{t_i}$.

Define thresholds $t_i$.

$r_i$ = Revenue from item pricing for $i$.

$= \max_p \; p(1 - F_i(p))$

$SRev = \sum_i r_i = r.$

Set $t_i = r$

Lemma 2: Suppose we partition the space of all value vectors into components $C_1, C_2, \ldots$

$$Rev(F) \leq \sum_j Pr[C_j] \cdot Rev(\underbrace{F_{|C_j}}_{\text{dist. over values conditioned on being in } C_j})$$

dist. over values conditioned on being in $C_j$.

For every $A \subseteq [m]$, $C_A = \left\{ v : \begin{array}{l} \forall i \in A,\ \underset{i \text{ is in tail}}{v_i > t_i} \\ \forall i \notin A,\ \underset{i \text{ is in Core}}{v_i \leq t_i} \end{array} \right\}$

$$F_A = F_{|C_A}$$

$$Rev(F) \underset{(\text{lemma2})}{\leq} \sum_A \Pr[\text{Tail} = A] \cdot Rev(F_A)$$

$$\leq \sum_A \Pr[A] \cdot \left\{ Rev(F_A^{\text{Tail}}) + Val(F_A^{\text{Core}}) \right\}$$

$$\leq \underbrace{\sum_A \Pr[A]\, Rev\left(F_A^{\text{Tail}}\right)} + Val(\text{Core})$$

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## Revenue from the Tail

In the unit-demand setting, $Rev(D) \leq 4\, SRev(D)$.

For some set $A$ and additive values over this set,

$$Rev(D_{|A}) \leq |A|\ Rev(\text{unit-demand buyer over } A)$$

$$\leq |A| \cdot 4\, SRev(D_{|A})$$

$$\sum_A \Pr[A]\, Rev(F_A^{\text{Tail}}) \leq 4\, SRev(F) \cdot E[|\text{Tail}|]$$

$$E[|\text{Tail}|] = \sum_i q_i \leq \sum_i \frac{r_i}{t_i} = \frac{\sum_i r_i}{SRev} = \frac{SRev}{SRev} = 1.$$

## Value from Core.

In the Core, $v_i \in [0, r]$

Claim : $\sum_i v_i$ is concentrated.

Strategy : Relate expectation of $\sum_i v_i$ to its median.

Lemma : $\text{mean of sum} \le C_1 \underbrace{\text{median of sum}}_{\text{BRev}} + C_2 \underbrace{r}_{\text{SRev}}$

subadditive function of values.

Revenue from pricing $i$ $= r_i$.

For any $p$, $\quad p\,(1 - F_i(p)) \le r_i$.

$$F_i(p) \ge 1 - \frac{r_i}{p}. \qquad f_i = \frac{r_i}{p^2}$$

$$\mathrm{Var}(v_i) \le E[v_i^2] \le \int_0^r t^2 \cdot \frac{r_i}{t^2}\,dt + r^2 \cdot \frac{r_i}{r}$$

$$= 2 r_i r$$

$$\mathrm{Var}\left(\sum_i v_i\right) \le \sum_i \mathrm{Var}(v_i) \le \sum_i 2 r_i r$$

$$\le 2r^2$$

$\mu = E\left[\sum_i v_i\right]$

$$\Pr\left[\sum_i v_i \le \frac{1}{4}\mu\right]$$

$$\leq \Pr\left[\left|\sum_i v_i - \mu\right| > \frac{3}{4}\mu\right]$$

$$\leq \frac{\mathrm{Var}(\sum_i v_i)}{(3\mu/4)^2}$$

$$\leq \frac{2r^2 \cdot 16}{9\mu^2} = \frac{32}{9} \cdot \frac{r^2}{\mu^2}.$$

Two cases:

① $\mathrm{SRev} \geq \frac{1}{4}\mu$ — done!

② $\mathrm{SRev} \leq \frac{1}{4}\mu$, $\Pr\left[\sum_i v_i \leq \mu/4\right] \leq \frac{32}{9} \cdot \frac{1}{16}$

$$= 2/9.$$

$$\mathrm{BRev} \geq \frac{\mu}{4} \cdot \frac{7}{9} \geq \text{const.}\ \mu.$$

$$\Rightarrow \max(\mathrm{SRev}, \mathrm{BRev}) \geq \frac{1}{4} \mathrm{Val}(\mathrm{Core}).$$

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$$\mathrm{Opt} \leq \mathrm{Rev}(\mathrm{Tail}) + \mathrm{Value}(\mathrm{Core})$$

$$\leq \text{const.}\ \mathrm{SRev} + \text{const.}\ \max(\mathrm{SRev}, \mathrm{BRev}).$$

$$\leq \text{const.}\ \max(\mathrm{SRev}, \mathrm{BRev}).$$

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## Extensions

— Single buyer setting with indep. item values extended subadditively to sets.

↓

[Rubinstein Weinberg '15]

Better of item & bundle pricing is approx optimal.

— Multiple buyers. — indep. item values extended subadditively to sets.

↓

[Chawla Miller '16]

[Cai Zhao '17]

Approx optimality through a Sequential Mechanism

— Offers to each buyer a two-part tariff.

(entry fee, per item prices)

↓ e.g. membership fee

↓ selling separately

Ex-ante Relaxation.

For a single buyer, define $Rev_{q_i}(F_i)$

$q_i = (q_{i1}, q_{i2}, \ldots)$

↓ max. rev. get

ex-ante supply constraint.

from buyer $i$
under constraint
that item $j$
is allocated to $i$
w.p. $\leq q_{ij}$.

$$\text{Rev} \leq \max_{\sum_i q_i \leq (1 \dots 1)} \sum_i \text{Rev}_{q_i}(F_i)$$

ex-ante
supply
constraint.