10/5/20.

- Review Myerson's mechanism
- Simplicity versus optimality
  - Vickrey auction with monopoly reserves.
    - Simplifying dependence on distributions.
  - Non-anonymous pricing
    - completely remove competition
  - Bulow-Klemperer result.
    - extra competition; remove dependence on dist.
  - Vickrey with anonymous reserve.
    - Prior-independent mechanisms.

- HW 1 due next Monday Oct 12.
- Info about projects coming soon.

Single parameter optimal mechanism design.

n buyers $\quad v_i \sim F_i$ — known.
Feasibility constraint,
Goal: BIC mechanism that maximizes revenue.

Myerson's lemma: BIC mechs $\Longleftrightarrow$ x is non-dec.
Payment identity

Theorem: Exp. revenue of any BIC mechanism is equal to the expected virtual surplus.

$$Rev(x) = \mathop{E}_{v}\left[\sum_i x_i \phi_i\right]$$

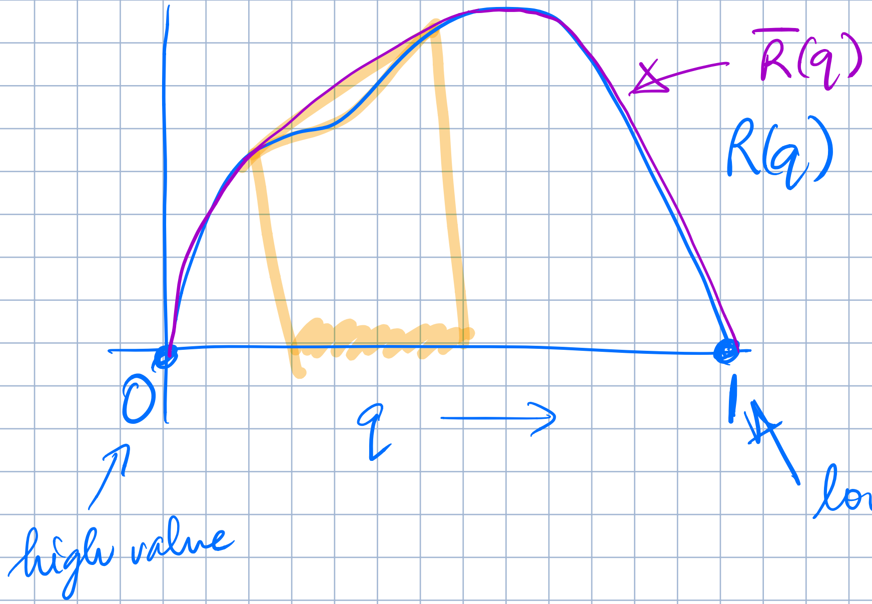
Virtual values: $\phi_i(v_i) = v_i - \dfrac{(1 - F_i(v_i))}{f_i(v_i)}$

Theorem: When all buyers have regular value distributions, pointwise maximization of virtual surplus gives a BIC mechanism.

$\therefore$ Optimal mech. is the virtual surplus maximizer.

Regularity $\equiv$ $\phi_i(\cdot)$ is a non-dec. function.

In the non-regular case, we need to iron.



$$R(q) = q\, F^{-1}(1-q)$$

$$\phi(q) = R'(q)$$

$$\bar{\Phi}(q) = \bar{R}'(q).$$

① Revenue of any BIC mech $\leq$ ironed virtual surplus.

② Ironed v.v. are non-decreasing.

③ Rev. of any BIC mech. that has a flat/constant allocation function over ironed regions
$=$ ironed virtual surplus.

$\Longrightarrow$ Revenue optimal mechanism is the ironed virtual surplus maximizer.

<u>Example</u>. 2 buyers & 1 item

$v_1 \sim U[0,1]$

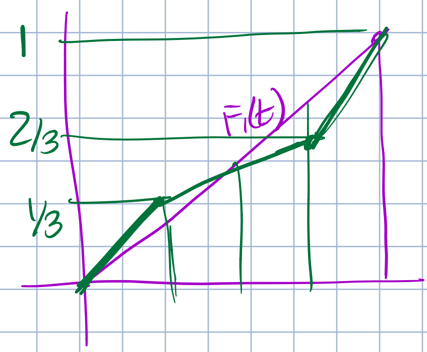
$F_1(t) = t$

$f_1(t) = 1$

$\phi_1(t) = 2t-1$

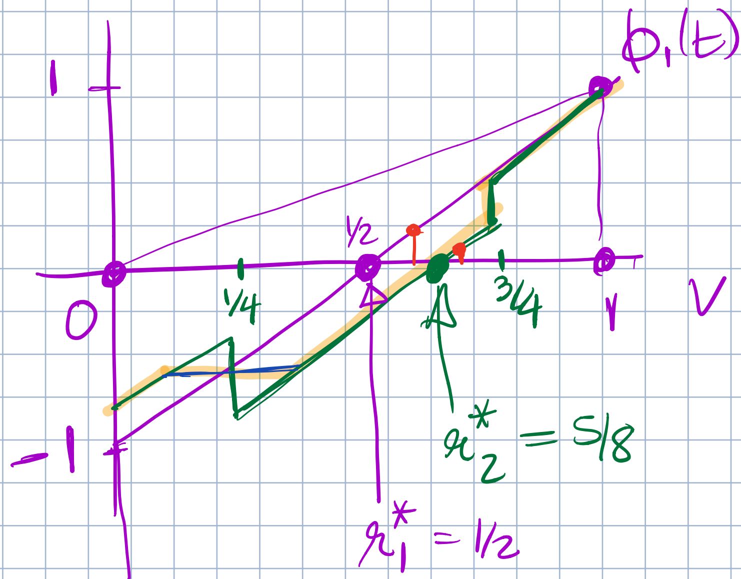
$V_2 \sim \frac{1}{3} U[0,\frac{1}{4}] + \frac{1}{3} U[\frac{1}{4},\frac{3}{4}] + \frac{1}{3} U[\frac{3}{4},1]$

$$F_2(t) = \begin{cases} 4t/3 & t \leq 1/4 \\ 2t/3 + 1/6 & t \in [1/4, 3/4] \\ 4t/3 - 1/3 & t \geq 3/4 \end{cases}$$



$$f_2(t) = \begin{cases} 4/3 & [0, 1/4] \cup [3/4, 1] \\ 2/3 & [1/4, 3/4] \end{cases}$$

$$\phi_2(t) = \begin{cases} 2t - 3/4 & t \leq 1/4 \\ 2t - 5/4 & [1/4, 3/4] \\ 2t - 1 & [3/4, 1] \end{cases}$$



Buyer 1 wins if:

- $v_1 \geq 1/2$ and

either $v_1 \geq 3/4$ and $v_1 \geq v_2$

or $v_1 \in [5/8, 3/4]$ and $v_2 \leq 3/4$

or $v_1 \in [1/2, 5/8]$ and $v_2 \leq v_1 + 1/8$

Vickrey with monop. reserves:

$v_1 \geq 1/2$ and $v_1 \geq v_2$

Rest of this lecture: Single item and regularity.

Showed last time: Vickrey with monopoly reserves gets a 2-approx to expected revenue.

Posted price Mechanism:

Given $F_1, \dots F_n$. Determine prices $p_1 \dots p_n$.

Buyers will arrive one at a time.

1st buyer with $v_i \geq p_i$ will get the item at price p_i.

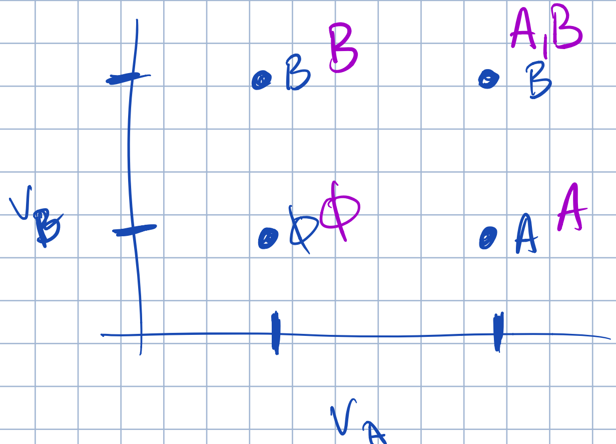
Sequential PM | Random order PM. | Order-oblivious PM.

Sequential PM $\downarrow$ specific "best case" ordering. impose an order

Random order PM $\downarrow$ uniformly random arrival.

Order-oblivious PM $\downarrow$ "worst case" ordering

Ex-Ante Relaxation.



Supply constraint: Allocate to at most one buyer at every value vector.

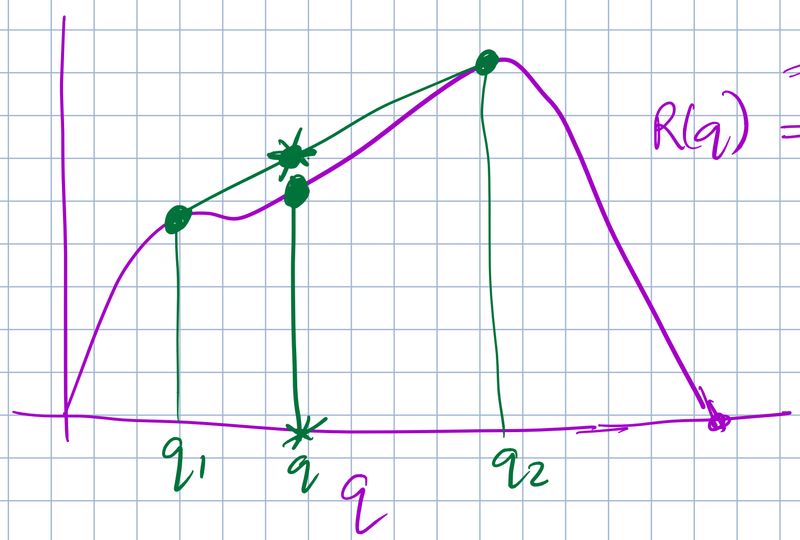
Ex-ante relaxation: Allocate at most one item in expectation over value vectors.

$Z_i(q_i)$ = Maximum revenue we can get from agent i by selling with ex-ante prob. at most q_i to the agent.

Ex-ante relaxation: $\bar{R}_i$

$$\text{EA-OPT} = \begin{cases} \max. & \sum_i Z_i(q_i) \\ \text{s.t.} & \sum_i q_i \le 1. \end{cases}$$

Lemma: Optimal revenue is at most EA-OPT.



$R(q)$ = Revenue from selling to agent at a price of $F^{-1}(1-q)$

$= q F^{-1}(1-q)$.

What is the best mech. that sells with ex ante prob q_i?

In the regular case, this corresponds exactly to posting a price of $F^{-1}(1-q)$.

$\Downarrow$

$Z(q) = R(q)$

In the non-regular case, $Z(q) = \bar{R}(q)$.

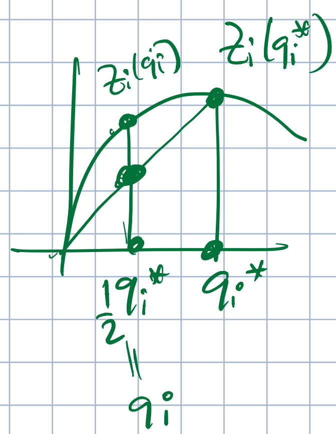
At quantiles q, where $R(q) \neq \bar{R}(q)$, the ex-ante optimal mech. randomizes between two posted prices.

OPM

Suppose $q_1^*, \dots, q_n^*$ is the solution to EA-OPT.

Set $q_i = \frac{1}{2} q_i^*$. and $p_i = F_i^{-1}(1-q_i)$. [Randomize in case of non-reg.]

Offer prices p_i in arbitrary sequence.



① $$\text{EA-OPT} = \sum_i z_i(q_i^*)$$

② $$\forall i, \quad z_i(q_i) \geq \frac{1}{2} z_i(q_i^*)$$

by concavity.

③ $$\sum_i q_i = \frac{1}{2} \sum_i q_i^* \leq \frac{1}{2}$$

Total prob. of selling the item is $\leq 1/2$

For any agent, prob. that item is still available when they arrive is $\geq 1/2$

④ $$\text{Rev of OPM} \geq \sum_i \Pr\left[\begin{array}{c}\text{item is offered to}\\ \text{agent } i\end{array}\right] \cdot \left(\begin{array}{c}\text{Rev from}\\ \text{agent } i\end{array}\right)$$

$$\geq \sum_i \frac{1}{2} \cdot z_i(q_i)$$

$$\geq 1/4 \sum_i z_i(q_i^*) = \frac{1}{4} \text{EA-OPT}$$

$\geq \frac{1}{4}$ OPT.

PROPHET INEQUALITY

$\boxed{X_1}$ $\boxed{X_2}$ $\boxed{X_3}$ . . . . . $\boxed{X_n}$

n boxes with a reward in each.

Alg. can open boxes one at a time,
observe the reward,
decide to accept or reject it.
$\Downarrow$
game ends.

Prophet knows the instantiations of all rewards.
So can accept the highest one.
Alg competes against $\mathbb{E}_{X_1..X_n}\left[\max_i X_i\right] = \text{OPT}$

Theorem: Can achieve ½ of prophet's expected reward by picking a single threshold and accepting the 1st reward that crosses it.

$$\text{Alg} = \mathbb{E}_{X_1..X_n}\left[X_{\underset{\text{any}}{\min}\{i \,:\, X_i \geq t\}}\right]$$

$\boxed{\text{Alg} \geq \tfrac{1}{2}\,\text{OPT}}$ for an appropriate choice of t go.

Connection to OPM's : the X's are virtual values;

OPT = virtual value maximizer

Alg = OPM that places a price of $\phi_i^{-1}(t)$ for buyer i.

Proof of prophet inequality

$$\Pr\left[\exists i : X_i \geq t\right] = 1/2$$

$$t = \text{median}\left(\max_i X_i\right)$$

$$\text{OPT} = E\left[\max_i X_i\right] \leq t + E\left[\max_i X_i - t\right]$$

$$\leq t + \sum_i (X_i - t)^+$$

$$\left[a^+ = \max(a, 0)\right]$$

$$\text{ALG} = \frac{1}{2}t + \sum_i \underbrace{\Pr\left[\begin{array}{l}\text{did not accept} \\ \text{a reward} \\ \text{before } i\end{array}\right]} (X_i - t)^+$$

$$\Pr\left[X_1 \ldots X_{i-1} < t\right]$$

$$= \Pr\left[\max_{i' < i} X_{i'} < t\right]$$

$$\geq \Pr\left[\max_i X_i < t\right]$$

$$= 1/2$$

$$\geq \frac{1}{2}t + \sum_i \frac{1}{2}(X_i - t)^+ \geq \frac{1}{2}\text{OPT}.$$

Non-anonymous posted pricing — different buyers are offered different prices

Approximation through increased competition.

Bulow-Klemperer

n i.i.d. buyers with regular values.

Optimal mechanism — Vickrey auction with anon. reserve.

✓ The revenue of the Vickrey auction over $(n+1)$ buyers (with no reserve price) is at least as large as revenue of optimal auction over n buyers.

Proof: Vickrey auction over $(n+1)$ buyers sells to the buyer with highest v.v. — even when this is −ve.

Among mechanisms that always sell the item over $(n+1)$ buyers, Vickrey is the optimal one.

Rev = OPT_n [Alternate mech over $n+1$ buyers:
- Run optimal mech over n buyers.
- If item remains unsold, give it for free to the $(n+1)^{th}$ buyer.

$$\text{Vickrey}_{n+1} \geq OPT_n.$$

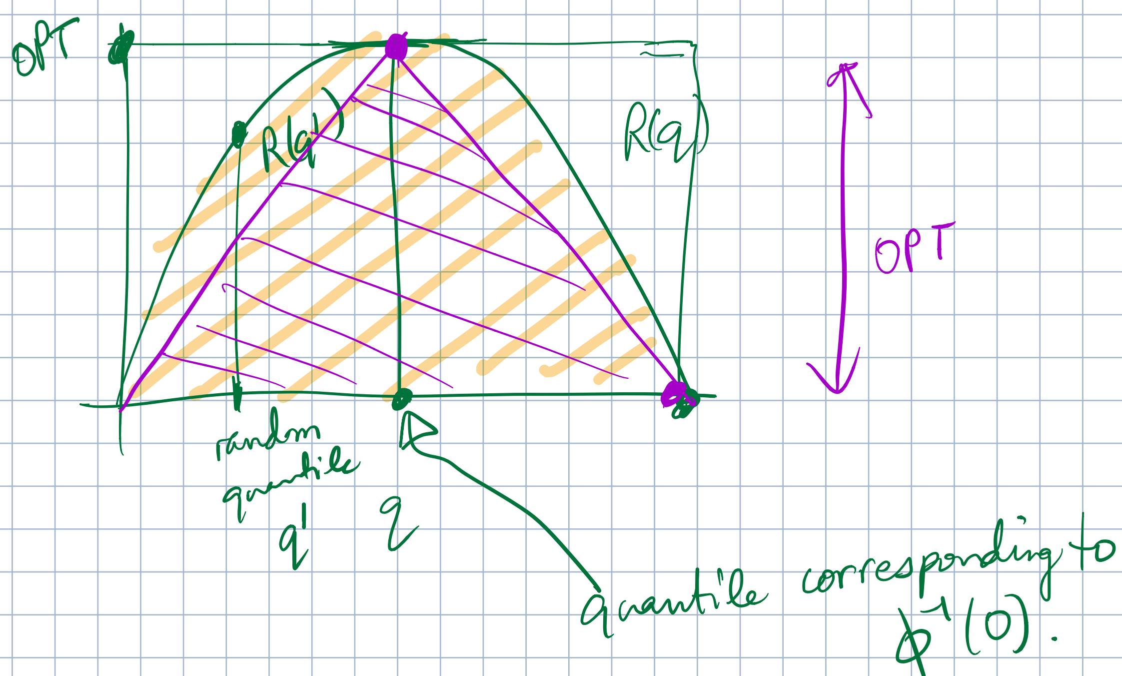
Single Sample Mechanisms. [DRY '10]

one buyer $v \sim F$. Optimal mechanism — offer $\phi^{-1}(0)$.

Draw $p \sim F$. Offer p to the buyer.

Claim: $\mathop{E}_{p \sim F}\left[Rev(p) \right] \geq \frac{1}{2}\, OPT$ if F is regular.

Proof.



$$\mathop{E}_{q' \sim U[0,1]}\left[R(q') \right] = \boxed{\int_0^1 R(q')\, dq'}$$

= area under curve

$\geq$ area of purple Δ.

Purple Δ has area $1/2$ OPT

and lies below $R(q)$ $\because$ R is concave.

[Hartline Roughgarden '09]

Consider n non-iid regular buyers. $v_1 \dots v_n$

Suppose we double the buyers. $v_1' \dots v_n'$.

Revenue of the Vickrey auction over $v_1 \dots v_n \, v_1' \dots v_n'$ is at least $1/2$ of the opt. rev. over $v_1 \dots v_n$.