Review Myerson’s mechanism
- Simplicity versus optimality
  - Vickrey auction with monopoly reserve
    - Simplifying dependence on distributions
  - Non-anonymous pricing
    - Completely remove competition
  - Bulow-Klemperer result
    - Extra competition; remove dependence on distribution
  - Vickrey with anonymous reserve
    - Prior-independent mechanism
  
- HW 1 due next Monday, Oct 12
- Info about projects coming soon
Single parameter optimal mechanism design.

- $n$ buyers $v_i \sim F_i$ — known.
- Feasibility constraint,
- Goal: BIC mechanism that maximizes revenue.

**Myerson's lemma**: BIC mechs $\iff$ $x$ is non-dec.

**Theorem**: Exp. revenue of any BIC mechanism is equal to the expected virtual surplus.

$$\text{Rev}(x) = \mathbb{E} \left[ \sum_i x_i \phi_i \right]$$

Virtual values: $\phi_i(v_i) = v_i - \frac{(1 - F_i(v_i))}{f_i(v_i)}$

- **Theorem** 2: When all buyers have regular value distributions, pointwise maximization of virtual surplus gives a BIC mechanism.
- Optimal mech. is the virtual surplus maximizer.

**Regularity**: $\phi_i(\cdot)$ is a non-dec. function.

In the non-regular case, we need to iron.
Revenue of any BIC mech. \leq \text{ironed virtual surplus}.

Ironed v.v. are non-decreasing.

Rev. of any BIC mech. that has a flat/constant allocation function over ironed regions = ironed virtual surplus.

\Rightarrow \text{Revenue optimal mechanism is the ironed virtual surplus maximizer.}

\underline{Example, 2 buyers & 1 item}

\begin{align*}
v_1 & \sim U[0, 1] \\
F_1(t) & = t \\
\Phi_1(t) & = 2t - 1
\end{align*}

\begin{align*}
v_2 & \sim \frac{1}{3} U[0, 1] + \frac{1}{3} U[\frac{3}{4}, 1] + \frac{1}{3} U[\frac{1}{2}, 1] \\
F_2(t) & = \begin{cases}
\frac{4}{3}t + \frac{1}{3} & t \leq \frac{1}{4} \\
\frac{2}{3}t + \frac{1}{6} & t \leq \frac{3}{4} \\
4 + \frac{1}{2} - V_2 & t \geq \frac{3}{4}
\end{cases}
\end{align*}
Buyer 1 wins if:
- \( u_1 \geq \frac{1}{2} \) and
  - either \( u_1 \geq \frac{3}{4} \) and \( u_1 \geq u_2 \)
  - or \( u_1 \in \left[ \frac{3}{8}, \frac{3}{4} \right] \) and \( u_2 \leq \frac{3}{4} \)
  - or \( u_1 \in \left[ \frac{1}{2}, \frac{3}{8} \right] \) and \( u_2 \leq u_1 + \frac{1}{8} \)
Rest of this lecture: Single item and regularity.

Showed last time: Vickrey with monopoly reserves gets a 2-approx to expected revenue.

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**Posted price Mechanism**

Given $F_1, \ldots, F_n$. Determine prices $p_1, \ldots, p_n$. Buyers will arrive one at a time.

1st buyer with $v_i \geq p_i$ will get the item at price $p_i$.

Sequential PM \quad \mid \quad Random order PM \quad \mid \quad Order-oblivious PM

\text{specific "best case" ordering.} \quad \updownarrow \quad \text{uniformly \ "worst case" ordering.}

\text{impose an order} \quad \mid \quad \text{random arrival.}

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**Ex-Ante Relaxation**

Supply constraint: Allocate to at most one buyer at every value vector.

Ex-ante relaxation: Allocate at most one item in expectation over value vectors.
Z_i(q_e) = Maximum revenue we can get from agent \(i\) by selling with ex-ante prob. at most \(q_e\) to the agent.

Ex-ante relaxation: \[ \frac{R_i}{E_A^{OPT}} = \max \sum_{q_e} Z_i(q_e) \]

subject to \[ Z_i(q_e) \leq 1 \]

Lemma: Optimal revenue is at most \(E_A^{OPT}\).

\[ R(q) = \text{Revenue from selling to agent at a price of } F^{-1}(1-q) \]

What is the best mech. that sells with ex-ante prob. \(q_e\)?

In the regular case, this corresponds exactly to posting a price of \(F^{-1}(1-q)\).

\[ E(q) = R(q) \]

In the non-regular case, \(E(q) = \bar{R}(q)\).

At quantiles \(q\), where \(R(q) + \bar{R}(q) = 1\), the
Suppose $q_1^*, \ldots, q_n^*$ is the solution to $\text{EA-OPT}$.

Set $q_i = \frac{1}{2} q_i^*$, and $p_i = \mathbb{P}_i^d(1-q_i)$. Randomize in case of non-reg.

Offer prices $p_i$ in arbitrary sequence.

1. $\text{EA-OPT} = \sum_i \mathbb{Z}_i(q_i^*)$

2. Hence, $\mathbb{Z}_i(p_i) \geq \frac{1}{2} \mathbb{Z}_i(q_i^*)$ by concavity.

3. $\sum_i q_i = \frac{1}{2} \sum_i q_i^* \leq \frac{1}{2}$

   Total prob. of selling the item is $\leq \frac{1}{2}$

   For any agent, prob. that item is still available when they arrive is $\geq \frac{1}{2}$

4. $\text{rev of } \text{OPM} \geq \sum_i \mathbb{P}_i^d(\text{item is offered to agent } i) \mathbb{P}_i \mathbb{Z}_i(q_i) \geq \frac{1}{2} \sum_i \mathbb{Z}_i(q_i^*)$
PROPHET INEQUALITY

\[
\begin{array}{cccc}
X_1 & X_2 & X_3 & \cdots & X_n \\
\end{array}
\]

There are \( n \) boxes with a reward in each.

Alg. can open boxes one at a time, observe the reward, decide to accept or reject it. Game ends.

Prophet knows the instantiations of all rewards.
So can accept the highest one.

Alg. competes against 
\[ E \left( \max_{x \in \mathbb{X}} x \right) = \text{OPT} \]

Theorem: Can achieve \( \frac{1}{2} \) of prophet's expected reward by picking a single threshold and accepting the 1st reward that crosses it.

\[
\text{Alg} = \mathbb{E}_{x_1 \cdots x_n} \left( \min_{i \in [n]} : x_i \geq t \right) \\
\]

\[ \text{Alg} \geq \frac{1}{2} \text{OPT} \] for an appropriate choice of \( t \).
Connection to OPM's 8's are virtual values \( \Phi_i \).

\( \text{OPT} = \) virtual value maximizer

\( \text{Alg} = \) OPM that places a price of \( \Phi_i^+ (t) \)

for buyer \( i \).

\[ \Pr \left[ \exists i : x_i \geq t \right] = 1/2 \]

\[ t = \text{median} \left( \max_i x_i \right) \]

\[ \text{OPT} = \mathbb{E} \left[ \max_i x_i \right] \leq t + \mathbb{E} \left[ \max_i x_i - t \right] \]

\[ \leq t + \sum \left( x_i - t \right)^+ \]

\[ [a^+ = \max (a, 0)] \]

\[ \text{Alg} = \frac{1}{2} t + \sum \Pr \left[ \text{did not accept a reward before } i \right] (x_i - t)^+ \]

\[ \geq \frac{1}{2} t + \sum \frac{1}{2} (x_i - t)^+ \geq \frac{1}{2} \text{OPT}. \]
Approximation through increased competition.

\[ \text{The revenue of the Vickrey auction over } (n+1) \text{ buyers with no reserve price is at least as large as revenue of optimal auction over } n \text{ buyers.} \]

\[ \text{Proof:} \] Vickrey auction over \((n+1)\) buyers sells to the buyer with highest \(v\) - \(v\) even when this is \(-v\).
Among mechanisms that always sell the item over \((n+1)\) buyers, Vickrey is the optimal one.

\[ \text{Alternate mech over } n+1 \text{ buyers:} \]

\[ \begin{align*}
\text{Rev} &= \text{OPT}_{n+1} \\
\text{OPT}_n &= \text{OPT}_n
\end{align*} \]

\[ \text{Vickrey}_{n+1} \geq \text{OPT}_n. \]
Single Sample Mechanism.  \cite{DRY10}\\
one buyer $1 \sim F$.  Optimal mechanism - offer $\Phi^{-1}(0)$.\\
Draw $\rho \sim F$.  Offer $\rho$ to the buyer.\\
\textbf{Claim}: $E[\text{Rev}(\rho)] \geq \frac{1}{2} \text{OPT}$ if $F$ is regular.\\

\textbf{Proof}: $\text{OPT}$\\

\begin{align*}
E_{\rho \sim UF} \left[ R(q') \right] &= \int_0^1 R(q') \, dq' \\
&= \text{area under curve} \\
&\geq \text{area of purple } \Delta.
\end{align*}

Purple $\Delta$ has area $\frac{1}{2} \text{ OPT}$\\
and lies below $R(q')$.  $R$ is concave.
Consider $n$ non-iid regular buyers: $U_1, \ldots, U_n$. Suppose we double the buyers: $U'_1, \ldots, U'_n$. Revenue of the Vickrey auction over $U_1, \ldots, U_n, U'_1, \ldots, U'_n$ is at least $\frac{1}{2}$ of the opt. rev. over $U_1, \ldots, U_n$. 