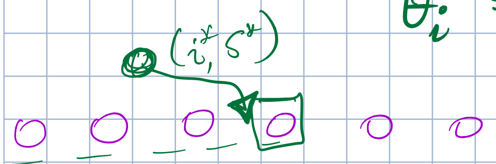
What about other kinds of value functions?

'2011

<u>A greedy algo</u> by Borodin & Lucier:

- Ask agents to report $\tilde{v}_i(\cdot)$.
- While items remain unallocated,
  - let $(i, X_i) = \underset{(i,S)\,:\,S \text{ is unallocated}}{\operatorname{argmax}} \; f_i(\tilde{v}_i, S)$
  - Allocate X_i to i.
- Charge each agent its critical price.

$$\theta_i = \begin{array}{l}\text{min reported value } \tilde{v}_i(X_i) \\ \text{at which } i \text{ would win } X_i.\end{array}$$



Define $\theta_{i,S}$ = min value $\tilde{v}_i(S)$ that i needs to report for S to be allocated S.

No longer DSIC.

Assumption: Underlying greedy alg. is c-approx
f_i is non-dec. in $\tilde{v}_i$ and non-inc. in S

NASH EQUILIBRIUM

A tuple of strategies $(\sigma_1, \cdots, \sigma_n)$ forms a Nash Equil if for all agents $i \in [n]$, fixing σ_{-i}, agent i's best response is to play σ_i.

[Multiple NE may exist.]

Claim: Suppose that the functions f_i are monotone non-decreasing in v_i and the above greedy algorithm is a c-approximation.
Then, in any NE of the greedy mechanism, the SW obtained is a $(c+1)$-approx. to the optimal SW.

$$\boxed{PoA \leq c+1.}$$

1. Observation: bids $\leq$ values.

$$\sum_{\substack{(i,x_i) \text{ allocated} \\ \text{by greedy}}} v_i(x_i) \geq \sum_{\substack{(i,x_i) \text{ alloc.} \\ \text{by greedy}}} \tilde{v}_i(x_i)$$

2. $$\sum_{(i,x_i)} \tilde{v}_i(x_i) \geq \frac{1}{c} \underbrace{\sum_{(i,S_i)\in OPT} \theta_i(S_i)}$$

3. Apply NE property.

$$\overbrace{\sum_{(i,S_i)\in OPT} v_i(S_i)}$$

2. $$\sum_{(i,x_i)\in \text{Greedy}} \tilde{v}_i(x_i) \geq \frac{1}{c} \sum_{\substack{(i,S_i) \text{ in} \\ \text{any feasible} \\ \text{solution}}} \tilde{v}_i(S_i)$$

$\sim$ OPT

For any $\epsilon > 0$.
Imagine changing the bid for each $(i,S_i) \in OPT \setminus \text{Greedy}$ to $\theta_i(S_i) - \epsilon$. Then Greedy solution doesn't change.

Then, $\sum_{(i,x_i)\in Greedy} \tilde{v}_i(x_i) \geq \frac{1}{c}\left\{ \sum_{(i,S_i)\in OPT\setminus Greedy} (\theta_i(S_i) - \varepsilon) + \sum_{(i,S_i)\in OPT\cap Greedy} \theta_i(S_i) \right\}$

$$\geq \frac{1}{c} \sum_{(i,S_i)\in OPT} \theta_i(S_i) - \varepsilon \text{ times something.}$$

PRICE OF ANARCHY (maximization objective)

$$\max_{\text{Nash Equil.}} \frac{\text{Optimal obj.}}{\text{Objective at NE}}$$

Ratio. (like approx ratio)
Worst case over all possible equilibria.

So far, $\sum_{(i,x_i)\in Greedy} v_i(x_i) \geq \frac{1}{c} \sum_{(i,S_i)\in OPT} \theta_i(S_i)$

Agent i's strategy is a best response to others' bids in the NE we're analyzing.

Consider alternate strategy where i bids $v_i(S_i)$ on S_i

Util from new strategy $= v_i(S_i) - \theta_i(S_i)$

$$\leq v_i(x_i) - \theta_i(x_i)$$

$$\sum_{(i,S_i)\in OPT} v_i(S_i) \leq \sum_{(i,S_i)\in OPT} \theta_i(S_i) + \sum_{(i,x_i)\in} v_i(x_i)$$

$$\leq c \sum_{(i,x_i) \in \text{Greedy}} v_i(x_i) + \sum_{(i,x_i) \in \text{Greedy}} v_i(x_i)$$

$$\boxed{OPT \leq (c+1)\,\text{Greedy}}$$

Alternate approach : Maximal in Distributional Range.

SWM on set $\mathcal{F}$

- Given feasible set $\mathcal{F}$ and value functions $v_i : \mathcal{F} \to \mathbb{R}^+ \; \forall i \in [n]$
- Find $\underset{j \in \mathcal{F}}{\operatorname{argmax}} \sum_{i \in [n]} v_i(j)$.

Maximum In Range mechanism (MIR)

Want: OPT over $\mathcal{F}'$ is approx-OPT over $\mathcal{F}$

- Find $\mathcal{F}' \subseteq \mathcal{F}$ such that SWM over $\mathcal{F}'$ is "easy".
  computationally easy
- Run VCG over $\mathcal{F}'$.

Maximum In Distributional Range mechanism (MIDR)

- Find $\mathcal{F}' \subseteq \Delta^{\mathcal{F}}$ such that SWM over $\mathcal{F}'$ is "easy".
- Run VCG over $\mathcal{F}'$.

Distributions over outcomes

Lavi-Swami '05 approach

n agents, m items, $\mathcal{F}$ = all partitions of items across agents.

Configuration LP.
(Relaxation)

x_{iS} : set S is allocated to i

$$\max \sum_{i,S} x_{iS} \, v_i(S)$$

s.t.

$$\sum_i \sum_{S \ni j} x_{i,S} \le 1 \qquad \forall j \in [m]$$

$$\sum_S x_{iS} \le 1 \qquad \forall i \in [n]$$

$$x_{i,S} \ge 0 \qquad \forall i \in [n], j \in [m]$$

Claim: Config-LP can be solved exactly in polynomial time.
Using a sep. oracle for the dual.
But also, need only demand queries to solve dual
Ask an agent to pick fav. set at given prices

Theorem: [LS'05] Suppose there is a c-approximate rounding algorithm for the Config-LP. Then given any x feasible for the LP, we can find a distribution D over deterministic partitions such that $SW(D) = \frac{1}{c} \cdot SW(x)$.

Write $\frac{1}{c} \cdot x$ as a convex combination of many deterministic partitions] integral solution.

alternate way of getting a c-approx.

Let $ALG(x) := $ dist D as output above.

$$\mathcal{F}' = \{ ALG(x) : x \in \text{feasible set of Config-LP} \}.$$

Mech:

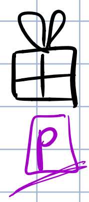
- Solve for x^* — opt. to Config LP.
- Return $ALG(x^*)$.
- Use VCG prices.

observe: $ALG(x^*)$ is optimal over $\mathcal{F}'$.

Revenue Maximization.

Seller with some items
agents with values.

seller allocates &
charges prices



agent 1's bid = \$1
agent 2's bid = \$10.

- Can't easily compare to an incentive-free optimum

- What's a good alternative benchmark?

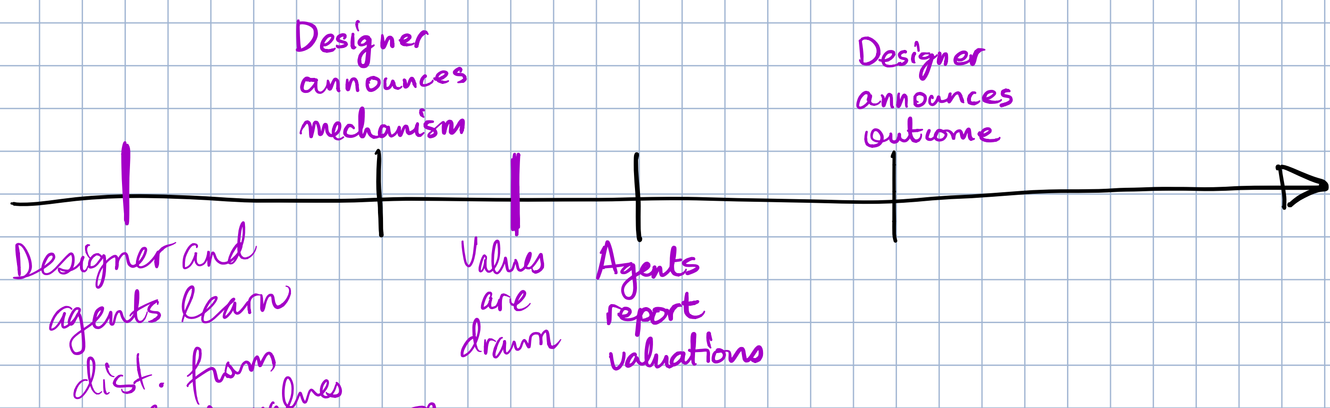
Goal: maximize total price charged

$$V_1 \sim \frac{1}{2}\,\$1 + \frac{1}{2}\,\$10$$

$$V_2 \sim \frac{1}{2}\,\$1 + \frac{1}{2}\,\$10$$

Bayesian assumption: Agent i's value function is drawn from a known distribution.

Timeline for direct revelation mechanisms.



which ... are drawn.

Simplest setting : Single item auction ; single agent. $v \sim F$

Prob. of allocation.

Outcome space : Allocation $x \in [0,1]$ Payment $p \in \mathbb{R}^+$

Agent's expected utility from outcome (x,p) = $vx - p$.

Mechanism designer's problem: Given distribution F develop functions $x(\cdot)$; $p(\cdot)$ such that

- (x,p) is DSIC / BIC
- $E_{v \sim F}[p(v)]$ is maximized.

Characterization of DSIC mechanisms

Myerson's Lemma (Special Case): For the single item, single agent problem, (x,p) is DSIC if and only if :

① x is monotone non-decreasing in v.

② – $\boxed{p(v)} = v\,x(v) - \boxed{\int_0^v x(t)\,dt} + p(0)$. Payment Identity

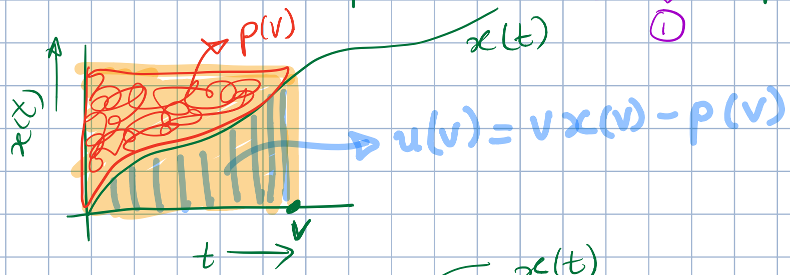
usually $p(0) = 0$

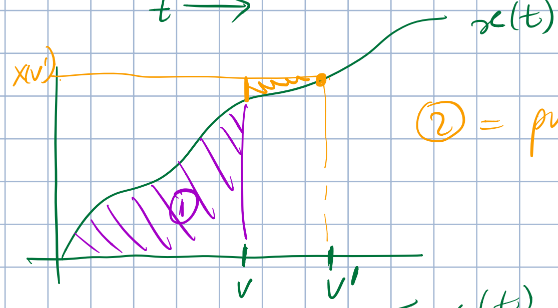
Proof:

$(\Leftarrow)$ Sps. (x,p) satisfy ① & ②.

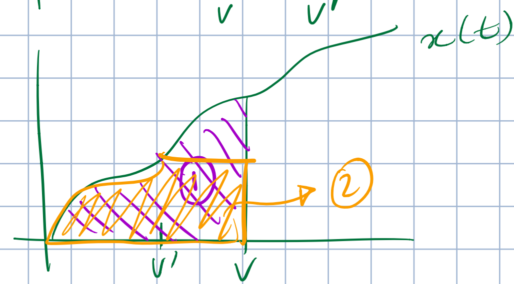
Consider an agent with value v and misreport v'.

Claim : $\overbrace{v\,x(v') - p(v')}^{②} \leq v\,x(v) - p(v)$.





② = purple minus orange.



$(\Rightarrow)$ $\forall v, v'$, $\boxed{v\,x(v') - p(v') \le v\,x(v) - p(v)}$

Also, $\boxed{v'\,x(v) - p(v) \le v'\,x(v') - p(v')}$

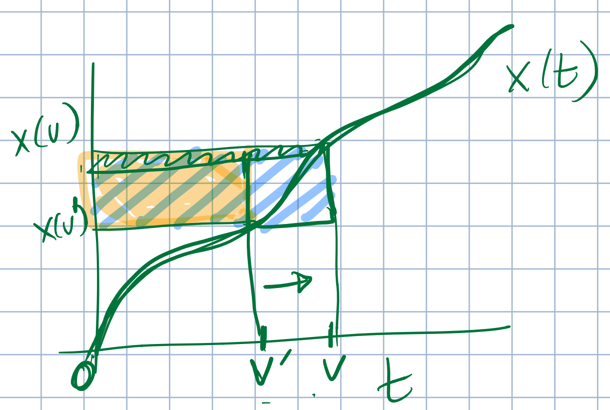
$$v\,x(v') + v'\,x(v) \le v\,x(v) + v'\,x(v')$$

$$(v - v')\,(x(v) - x(v')) \ge 0$$

$$\left.\begin{array}{l} \Rightarrow \text{If } v' < v \Rightarrow x(v') \le x(v) \\ \phantom{\Rightarrow \text{If }} v' > v \Rightarrow x(v') \ge x(v) \end{array}\right\} \text{monotonicity } ①$$

$p(v) - p(v') \le v\,(x(v) - x(v'))$

$p(v) - p(v') \ge v'\,(x(v) - x(v'))$



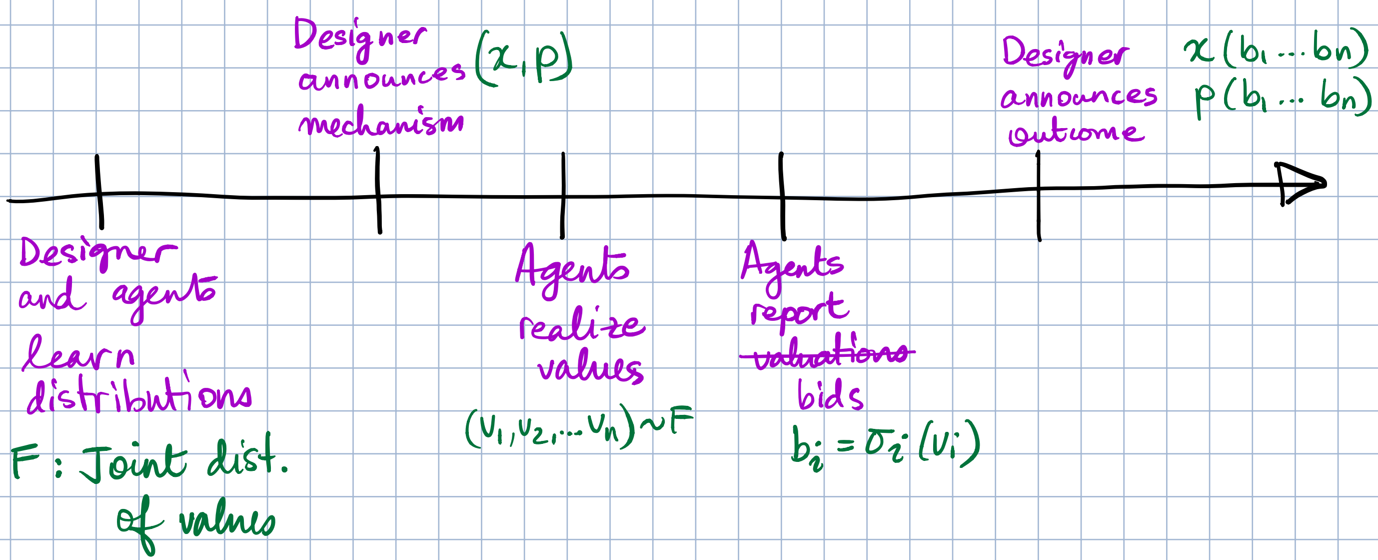
As $v' \to v^-$
the difference of payments
converges to area to the
right of $x(t)$.

linear in x_i.

General $\boxed{\text{Single}}$-parameter ($\boxed{\text{linear}}$ agents) setting

n agents ; "type" of each agent is given by one number $\boxed{v_i}$.
Outcome for each agent is $x_i \in [0,1]$; $p_i \in \mathbb{R}$.
Utility of linear agent i from (x_i, p_i) is $v_i x_i - p_i$.
Agents maximize expected utility.

Timeline for direct revelation mechanisms.



$x_i(b_1, \dots b_n)$ = alloc. to agent i
$p_i(b_1, \dots b_n)$ = payment of agent i.

Bayesian Incentive Compatibility (BIC).

A strategy σ_i for agent i maps values v_i to bids $\sigma_i(v_i)$.

A strategy tuple $(\sigma_1, \underbrace{\sigma_2, \cdots \sigma_n}_{\sigma_{-i}})$ is in **Bayes Nash Equilibrium** if for all agents i,

for all values v_i

for all alternate strategies σ_i'

Agent i knows $F|_{v_i}$ and σ_{-i}. but doesnot know values v_{-i}.

$$\mathop{E}_{\substack{v_{-i} \sim F|_{v_i} \\ b_{-i} = \sigma_{-i}(v_{-i})}} \left[v_i\, x_i(\sigma_i(v_i), b_{-i}) - p_i(\sigma_i(v_i), b_{-i}) \right]$$

$$\geq \mathop{E}_{\substack{v_{-i} \sim F|_{v_i} \\ b_{-i} = \sigma_{-i}(v_{-i})}} \left[v_i\, x(\sigma_i'(v_i), b_{-i}) - p(\sigma_i'(v_i), b_{-i}) \right]$$

A mechanism is BIC if truthtelling i.e. $\sigma_i(v_i) = v_i \ \forall i$ is a Bayes-Nash Equilibrium.

Assume : $F = F_1 \times F_2 \times \cdots \times F_n$. $\to$ agents have independent values.

Fix σ_{-i}.

Define $$x_i(b) = \mathop{E}_{\substack{v_{-i} \sim F|_{v_i} \\ b_{-i} = \sigma_{-i}(v_{-i})}} \left[x_i(b, b_{-i}) \right]$$

$$p_i(b) = \mathop{E}_{\substack{v_{-i} \sim F|_{v_i} \\ b_{-i} = \sigma_{-i}(v_{-i})}} \left[p_i(b, b_{-i}) \right]$$

Myerson's Lemma (General Case):

A mechanism (x, p) for the single parameter (linear agent) setting is BIC if and only if $\forall i$:

- $x_i(v_i)$ is monotone non-decreasing in v_i.
- $p_i(v_i) = v_i\, x_i(v_i) - \int_0^{v_i} x_i(t)\, dt + p_i(0)$

A mechanism (x, p) for the single parameter (linear agent) setting is DSIC if and only if $\forall i, v_{-i}$:

- $x_i(v_i, v_{-i})$ is monotone non-decreasing in v_i.
- $p_i(v_i, v_{-i}) = v_i\, x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(t, v_{-i})\, dt + p_i(0)$

x — p

x' — p'

$\frac{x+x'}{2}$ — $\frac{p+p'}{2}$

Implications of Myerson's Lemma :

① **Revenue Equivalence.**

Two mechanisms that have the same allocation rules will always have the same expected revenue.

e.g. Vickrey auction – highest value wins

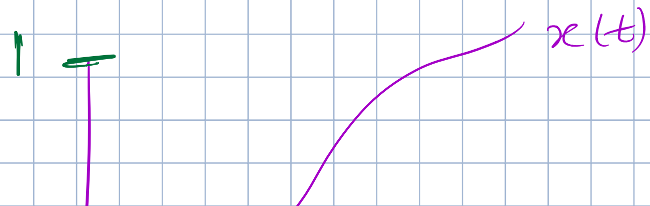
First Price sealed bid auction where every buyer's value is drawn from same dist. and so they use symmetric strategies.] highest value win

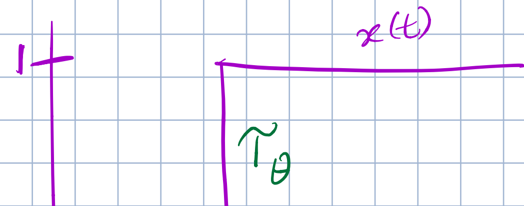
② **Revenue Linearity.**

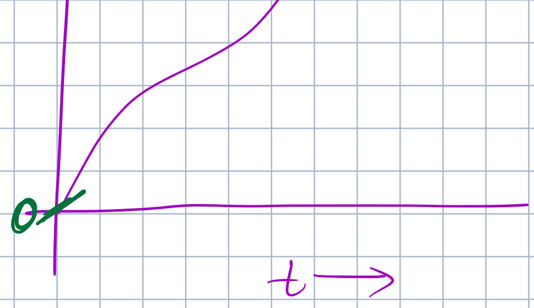
If we average the allocation rules of two mechanisms, then the expected revenue gets averaged.

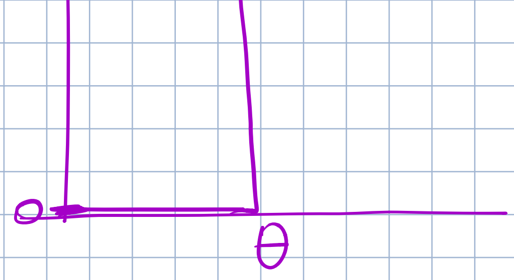
③ For the single parameter (linear agent) setting, the optimal BIC mechanism is deterministic. and DSIC.

Proof for the single agent setting

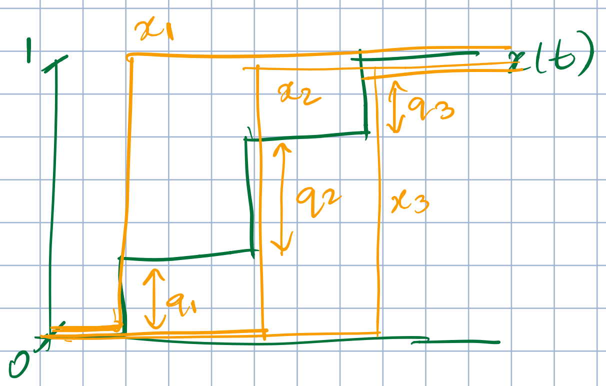








General mech.



$$x = q_1 x_1 + q_2 x_2 + q_3 x_3$$

Let $\tau_\theta(v) = \begin{cases} 1 & \text{if } v \geq \theta \\ 0 & \text{if } v < \theta \end{cases}$

For any non-dec. $x : \mathbb{R} \to [0,1]$

$$x(v) = \int_0^\infty x'(\theta)\, \tau_\theta(v)\, d\theta$$

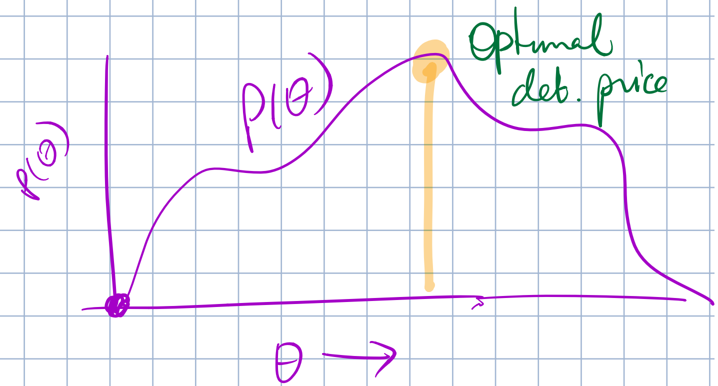
Det. mech. — pricing

Charges θ for allocating item

0 for not alloc.

$$\text{Revenue}(\theta) = \theta \cdot \Pr[v \geq \theta]$$

$$P(\theta) = \theta\,(1 - F(\theta))$$



Distribution over prices.

By revenue linearity,

$$\text{Rev}(x) = \int_0^\infty x'(\theta)\, \underbrace{\text{Rev}(\tau_\theta)}_{P(\theta)}\, d\theta$$

$$\max_{\text{monotone } x} \text{Rev}(x) = \max_{\text{prices } \theta} P(\theta)$$