REVENUE MAXIMIZATION.

A seller with one item;
several buyers with values for item.

Bayesian assumption : Agent i's value function is drawn from a Known distribution.

Mechanism designer's problem: Given distribution $F = F_1 \times F_2 \times \cdots \times F_n$.

develop functions $x(\cdot)$; $p(\cdot)$ such that

- (x, p) is DSIC / BIC
- $\sum_i E_{v \sim F}[p_i(v)]$ is maximized.

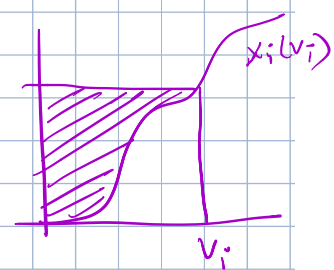
Single-parameter linear agents

- one value v_i
- served or not x_i
- util $= v_i x_i - p_i$.

Myerson's Lemma (General Case):

A mechanism (x, p) for the single parameter (linear agent) setting is BIC if and only if $\forall i$:

- $x_i(v_i)$ is monotone non-decreasing in v_i.
- $p_i(v_i) = v_i\, x_i(v_i) - \int_0^{v_i} x_i(t)\, dt + p_i(0)$



A mechanism (x, p) for the single parameter (linear agent) setting is DSIC if and only if $\forall i, v_{-i}$:

- $x_i(v_i, v_{-i})$ is monotone non-decreasing in v_i.
- $p_i(v_i, v_{-i}) = v_i\, x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(t, v_{-i})\, dt + p_i(0)$

Implication: Revenue is linear in x.

Agenda for today

1. Myerson's mechanism
   - Reduction from revenue maximization to virtual surplus max.
   - Ironing
   - Examples.

2. Simplicity versus optimality
   - Approximation using the Vickrey auction.
   - Approximation via increased competition
   - Approximation using anonymous reserve.

HW1 — out after lecture today
due in two weeks.

Rewrite the optimal mechanism design problem:

$$\max \sum_i \mathop{E}_{v_i \sim F_i} \left[p_i(v_i) \right]$$

s.t. ① $x_1, \dots, x_n$ are "feasible" [e.g. a supply constraint]

② $\forall i$, $x_i(\cdot)$ is weakly increasing

③ $\forall i$ $\quad p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x(t)\, dt$

[Assuming $p(0)=0$]

F_i : dist. of v_i

$F_i(v_i) = \Pr_{t \sim F_i} [t \leq v_i]$

$f_i(v_i)$ = density at v_i

$= F_i'(v_i)$

$$\mathop{E}_{v_i \sim F_i} \left[p_i(v_i) \right] = \int_0^\infty p_i(v_i) f_i(v_i) dv_i$$

$$= \int_0^\infty v_i x_i(v_i) f_i(v_i)\, dv_i - \int_0^\infty \left(\int_0^{v_i} x(t) dt \right) f_i(v_i) dv_i$$

$$= \int_0^\infty v x(v) f(v) dv - \int_{v=0}^\infty \int_{t=0}^{v} x(t) f(v)\, dt\, dv$$

$$= \int_0^\infty v x(v) f(v) dv - \int_{t=0}^\infty \underbrace{\left(\int_{v=t}^\infty f(v)\, dv \right)}_{\substack{F(\infty) - F(t) \\ = 1 - F(t)}} x(t)\, dt$$

$$= \int_0^\infty v x(v) f(v) dv - \int_{v=0}^\infty (1 - F(v))\, x(v)\, dv$$

$$= \int_0^\infty x(v) \left(v f(v) - (1 - F(v)) \right) dv$$

$$= \int_0^\infty x(v) \underbrace{\left(v - \frac{1-F(v)}{f(v)}\right)}_{\phi(v)} f(v)\, dv$$

$$= \int_0^\infty x(v)\,\phi(v)\, f(v)\, dv = \mathop{E}_{v \sim F}\left[x(v)\,\phi(v)\right]$$

Putting all agents together,

$$\max \sum_i \mathop{E}_{v_i \sim F_i}\left[x_i(v_i)\,\phi_i(v_i)\right] = \mathop{E}_{v}\left[\sum_i x_i(v_i)\,\phi_i(v_i)\right]$$

s.t. ① $x_1, \ldots, x_n$ are "feasible" [e.g. a supply constraint]

② $\forall i$, $x_i(\cdot)$ is weakly increasing

~~③ $\forall i$ $p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x(t)\, dt$~~

[Assuming $p(0)=0$]

$\phi_i(v_i)$ = "virtual" values.

Fix some value vector $v_1 \cdots v_n$ and alloc. $x_1 \ldots x_n$

SW of this allocation $= \sum_i x_i v_i$

Virtual SW $= \sum_i x_i \phi_i(v_i)$

Rev. Max $\iff$ Virtual SW maximization s.t. monotonicity of the x_i's

Approach :- Characterize optimum for relaxed problem where monot. constraint is removed.
- Figure out whether the constraint is satisfied.

Example 1 item. 2 agents

$v_1 \sim \text{Unif.}[0,1]$

$v_2 \sim \text{Unif.}[0,1]$

SW maximization $E[x_1 v_1 + x_2 v_2]$

- Agents report v_1, v_2
- Alloc. to 1 if $v_1 \geq v_2$: $x_1 = 1, x_2 = 0$
  2 if $v_2 > v_1$: $x_1 = 0, x_2 = 1$

$F_1(t) = t \quad f_1(t) = 1$

$F_2(t) = t \quad f_2(t) = 1$

Revenue max.

$$\max E[x_1 \phi_1 + x_2 \phi_2]$$

s.t. ~~x_1, x_2 non-dec.~~

one item alloc.

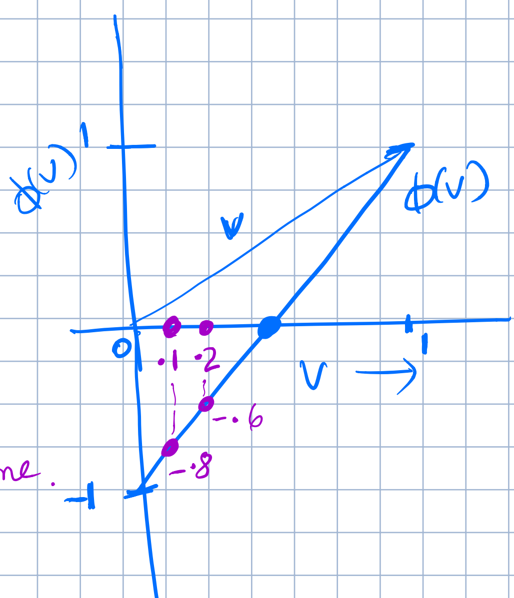
$$\phi(v) = v - \frac{1-F(v)}{f(v)} = v - \frac{(1-v)}{1} = 2v - 1$$

Point wise maximizes objective.

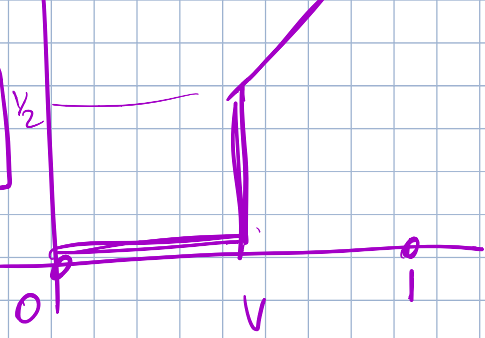
If $\phi_1 \geq \phi_2$ and $\phi_1 \geq 0$ then $x_1 = 1, x_2 = 0$

$\phi_2 > \phi_1$ and $\phi_2 \geq 0$ then $x_1 = 0, x_1 = 1$.

If $\phi_1, \phi_2 < 0$ then serve no one.



$$x_1(v_1) = \Pr_{v_2}\left[\phi(v_1) \geq 0 \text{ and } \phi(v_1) \geq \phi(v_2)\right]$$



$$= \begin{cases} 0 & \text{if } v_1 \leq \frac{1}{2} \\ \Pr[\phi_1 \geq \phi_2] & \text{if } v_1 > \frac{1}{2} \end{cases}$$

$$= \begin{cases} 0 & \text{if } v_1 \leq \frac{1}{2} \\ v_1 & \text{if } v_1 > \frac{1}{2} \end{cases}$$

Fixing v_1, what is $\Pr[\phi_2 < \phi_1]$?

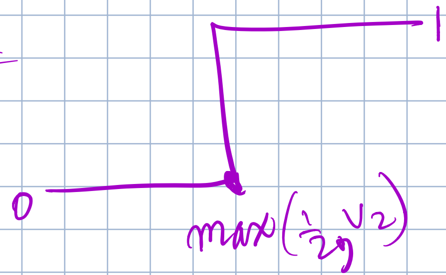
$$= \Pr_{v_2}[2v_2 - 1 < 2v_1 - 1]$$

$$= \Pr_{v_2}[v_2 < v_1] = v_1$$

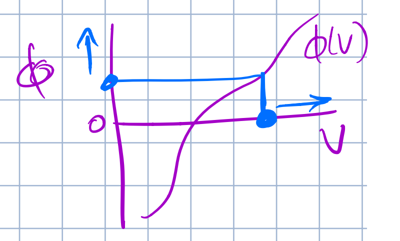
Observe: For any fixed v_2,

$$x_1(v_1, v_2) = \Pr\left[v_1 \geq \tfrac{1}{2} \text{ and } v_1 \geq v_2\right]$$

$=$



Regularity



A value distribution F is regular if $\phi(\cdot)$ is weakly non-decreasing.

Optimal mech : MYERSON'S MECHANISM

- Agents report $v_1 \ldots v_n$
- Compute $\phi_1, \ldots, \phi_n$.
- Return the feasible set that maximizes virtual surplus $\underset{S}{\text{argmax}}\left(\sum_{i \in S} \phi_i\right)$

Claim: If all value distributions are regular, then Myerson's mechanism is BIC.

Observe: deterministic ; DSIC

$$\phi(v) = v - \frac{1-F(v)}{f(v)}$$

$h(v) = \frac{f(v)}{1-F(v)}$ = hazard rate.

"Monotone Hazard Rate" (MHR) — $h(v)$ is non-dec.

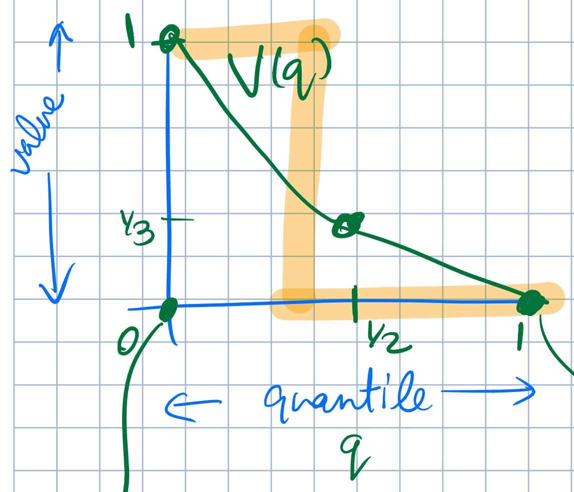
$\Rightarrow \phi(v)$ is non dec.

Examples to try on your own:

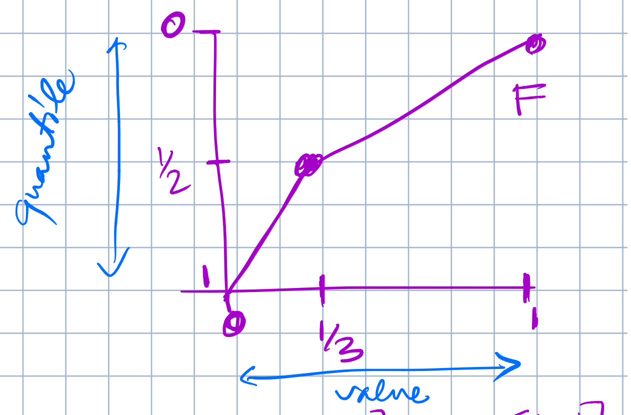
- Unif. $[a,b]$
- Exp. dist.
- Power law
- $f(v) = \frac{1}{v^2}$ for $v \in [1, \infty)$.

- Bimodal

$\frac{1}{2}$ Unif$[0,1]$ + $\frac{1}{2}$ Unif$[2,3]$



$\Longleftrightarrow$



low quantile ≡ high values.

high quantile ≡ low values.

$F = \frac{1}{2} U[0, \frac{1}{3}] + \frac{1}{2} U[\frac{1}{3}, 1]$

$q = 1 - F(v)$

$v = V(q)$

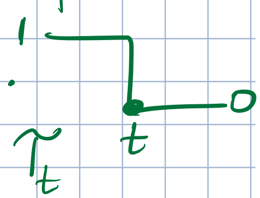
Focus on single agent.

$x(q)$ = alloc. to an agent with quantile q.

$Rev(x)$ = Expected revenue from alloc x.

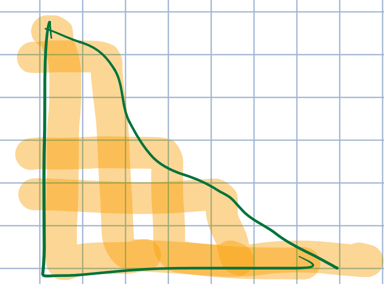
In quantile space, monotone allocs are dec. fns.

det. alloc. are step fns!



Rev Linearity

$$x(q) = -\int_t x'(t) \cdot T_t(q) \, dt$$



$$Rev(x) = -\int_{t=0}^{1} x'(t) R(t) \, dt$$

$$= \int_0^1 x(t) R'(t) \, dt - R(1)x(1) + R(0)x(0)$$

$R(t)$ = Revenue of T_t

$= V(t) \cdot t$

Note: $R(0) = 0 \therefore t = 0$

$R(1) = 0 \therefore V(t) = 0$

$$\int_a^b u'v = uv\Big|_a^b - \int v'u$$

For any weakly monotone alloc. function x,

$$Rev(x) = \int_q x(q) \underbrace{R'(q)}_{\text{Virtual value}} dq.$$

$\Rightarrow$ $\boxed{\text{Expected revenue} = \text{Expected Virtual Surplus}}$

$R(q) = q \cdot V(q)$. $\Rightarrow R'(q) = V(q) + q V'(q)$.

$V(q)$ is inverse of $1-F(v)$

$$\frac{d}{dq} V(q) = \frac{dv}{dq} = \frac{1}{dq/dv} = \frac{1}{(1-F(v))'} = -\frac{1}{f(v)}$$

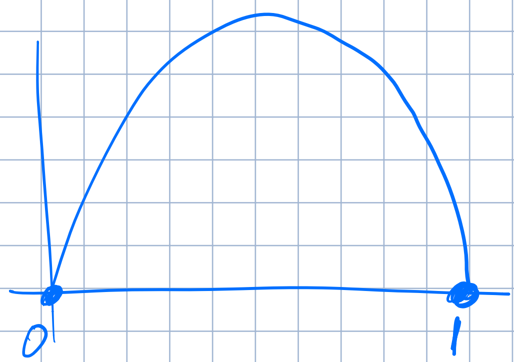
$$\boxed{\phi(v) = R'(q)} = v + 1-F(v)\left(-\frac{1}{f(v)}\right) = v - \frac{1-F(v)}{f(v)}.$$

Regularity: $\phi(v)$ is non-dec in v

$\phi(q) = \phi(V(q))$

$\Leftrightarrow \phi(q)$ is non-inc. in q.

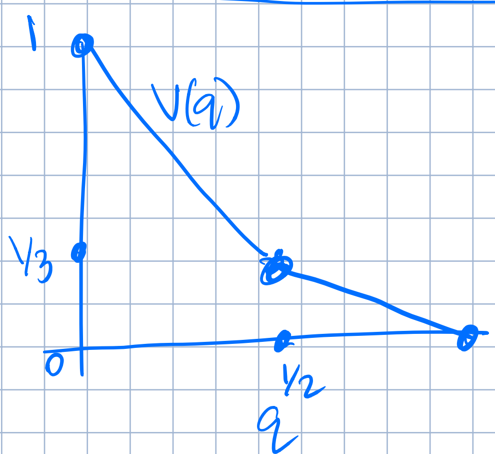
$\Leftarrow$ $R(q)$ is concave in q.



Exercise for home:

Plot $R(q)$ for

- Unif$[a,b]$
- Exp.
- Power law
- etc.



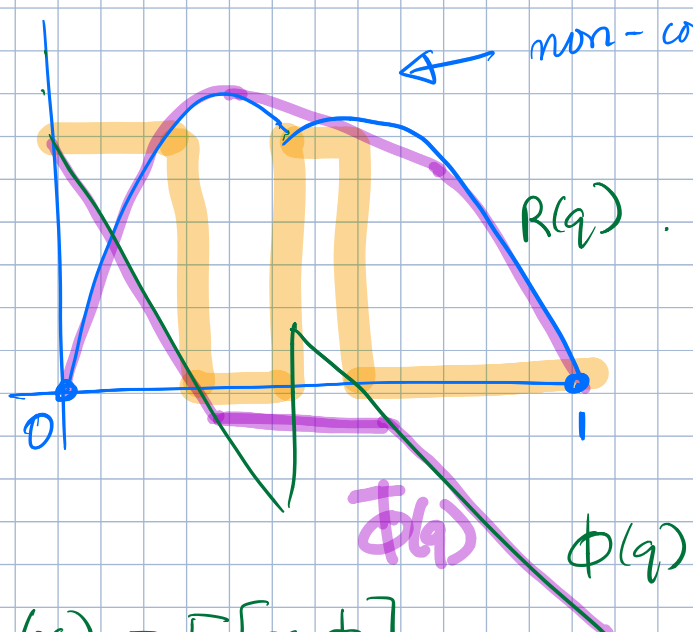
$$F \equiv \frac{1}{2} U[0, 1/3] + \frac{1}{2} U[1/3, 1].$$

$$R(q) = q\, V(q)$$

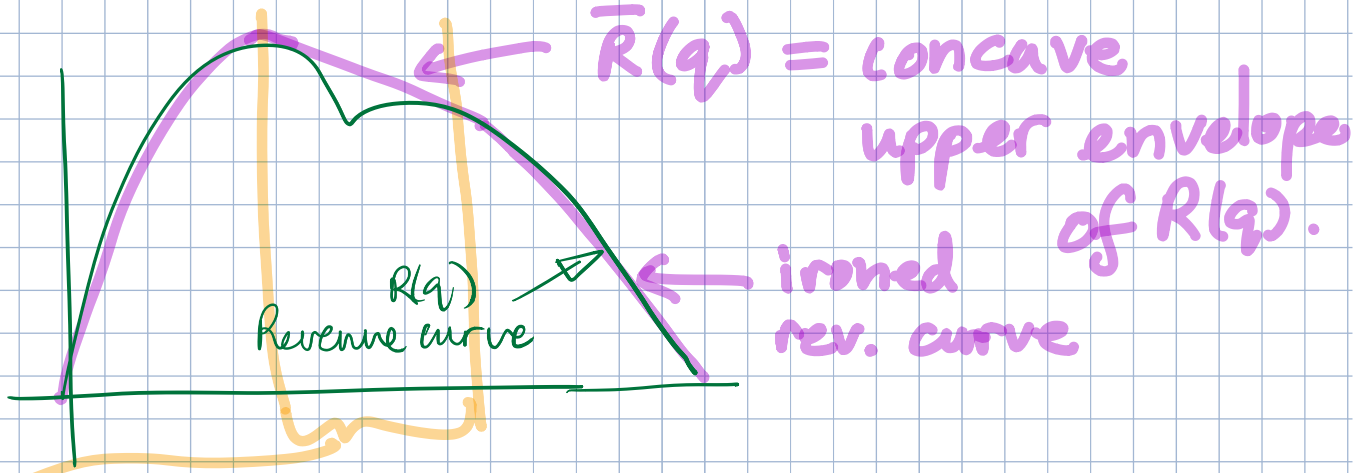
$$V(q) = \begin{cases} 1 - \frac{4}{3} q & q \le 1/2 \\ \frac{2}{3}(1-q) & q > 1/2 \end{cases}$$

$$R(q) = \begin{cases} q - \frac{4}{3} q^2 \\ \frac{2}{3} q - \frac{2}{3} q^2. \end{cases}$$

non-concavity.



$\mathrm{Rev}(x) = E[x \phi]$



Interval where $\bar{R}(q) \neq R(q)$, $\bar{R}(q)$ is linear $\Rightarrow \bar{\Phi}(q)$ is constant

$\bar{R}(q)$ = smallest concave fn. that lies above $R(q)$.

$\bar{\Phi}(q) = \bar{R}'(q)$.

Observe:

① $\bar{R}(q)$ is concave and so $\bar{\Phi}(q)$ is non-inc.

② $\mathrm{Rev}(x) = \underset{q}{E}[x(q)\,\phi(q)] = -\underset{q}{E}[x'(q)\,R(q)]$

If x is monotone (i.e. $x'(q) \leq 0 \;\forall q$)

then $\mathrm{Rev}(x) = -E[x'(q)\,R(q)] \leq -E[x'(q)\,\bar{R}(q)]$

$$= \underset{q}{E}[x(q)\,\bar{\Phi}(q)]$$

$\Rightarrow$ $\boxed{\text{Exp. revenue of } x \leq \text{ironed virtual surplus.}}$

③ For any mech. x such that ($x'(q) = 0$ whenever $R(q) \neq \bar{R}(q)$), (*)

virtual surplus = ironed virtual surplus.

(4) There is a monotone mech. maximizing ironed virtual surplus that satisfies (*)

MYERSON'S MECHANISM.

- Pointwise maximize ironed virtual surplus.
- If there are any ties in ivv, break ties consistently

Analysis:

Any interval where $\bar{R}(q) > R(q)$,

$\Rightarrow \bar{\Phi}(q)$ is constant

$\Rightarrow x(q)$ is constant

$\Rightarrow x'(q) = 0$

$\Rightarrow$ Ironed virtual surplus = Virtual surplus.

Example One item, two buyers.

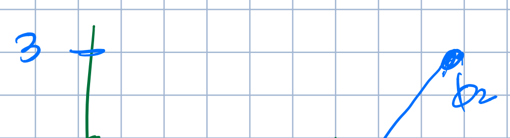
$F_1 = U[0,2] \qquad F_1(t) = t/2$

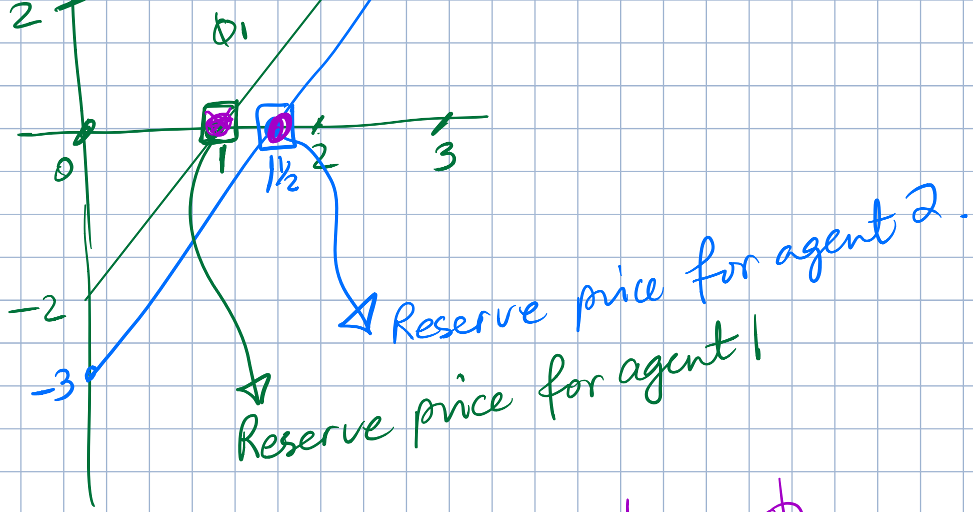
$F_2 = U[0,3] \qquad F_2(t) = t/3$

Regular

$$\phi_1(t) = t - \frac{1-F_1(t)}{f_1(t)} = t - \frac{1-t/2}{1/2} = 2t-2$$

$$\phi_2(t) = t - \frac{1-F_2(t)}{f_2(t)} = t - \frac{1-t/3}{1/3} = 2t-3$$





$$\phi_1 \geq \phi_2$$

$$\iff 2v_1 - 2 \geq 2v_2 - 3$$

$$\iff v_1 \geq v_2 - 1/2$$

Opt. Mech :

If $v_1 \leq 1$ reject agent 1

If $v_2 \leq 3/2$ reject agent 2.

If both remain, agent 2 needs to outbid 1 by $1/2$ to win.

Observe : Myerson's mechanism is discriminatory!.

If all the values are distributed i.i.d. & regular.

$\Rightarrow$ all v.v. fns. are identical.

$\Rightarrow$ agent with highest v.v. is also agent with highest value.

Monopoly reserve price $= \phi^{-1}(0) = r^*$.

Optimal mechanism for i.i.d. regular agents is

Vickrey auction with monopoly reserve

Simplicity versus Optimality

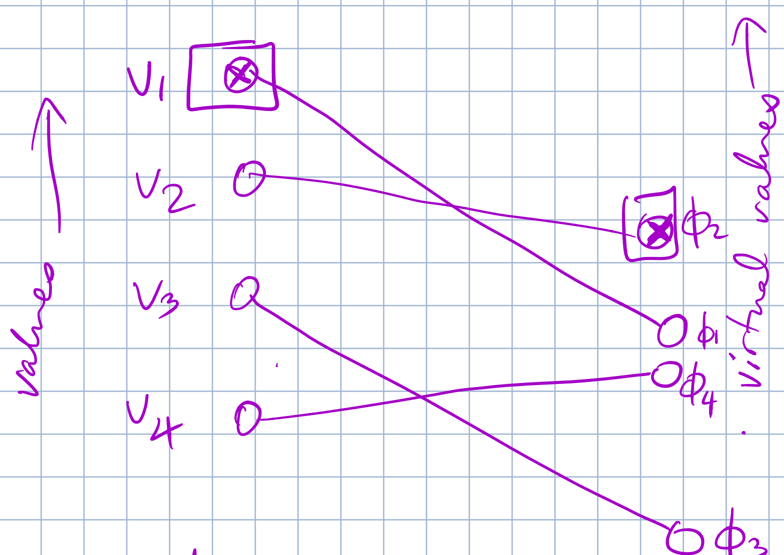
Approx. through Vickrey with (diff.) reserve prices.

Vickrey with monopoly reserves for non-iid settings:

- Compute $r_i^* = \phi_i^{-1}(0)$ $\forall i$.
- Reject all agents with $v_i < r_i^*$.
- Run Vickrey auction over the values reported. (virtual values). ← opt.
- Charge critical prices. ← $\max\left(r_i^*, \max_{\substack{j \neq i \\ v_j \geq r_j^*}} v_j\right)$

Theorem: In the non-iid regular setting, Vickrey with monopoly reserves gets a 2-approx. to exp. revenue. $\boxed{\text{TIGHT}}$

Proof:



i = winning agent in OPT

j = winning agent in Vickrey.

Opt. picked ϕ_2

Vickrey picked ϕ_1

Rev(Vickrey) = Exp. virtual surplus of Vickrey.

Note : $\forall i,\ \phi_i(v_i) \leq v_i$

$$\text{OptRev} = E[\phi_i] = \boxed{E[\phi_i \mid i=j]\, \Pr[i=j]} + \boxed{E[\phi_i \mid i\neq j]\, \Pr[i\neq j]}$$

$$E[\phi_i \mid i=j]\, \Pr[i=j] = E[\phi_j \mid i=j]\, \Pr[i=j]$$

$\phi_j \geq 0$

$$\leq E[\phi_j \mid i=j]\, \Pr[i=j] + E[\phi_j \mid i\neq j]\, \Pr[i\neq j]$$

$$= \text{Rev of Vickrey}$$

$$\text{Rev of Vickrey} = E[p_j \mid i\neq j]\, \Pr[i\neq j] + E[p_j \mid i=j]\, \Pr[i=j]$$

$$\geq E[v_i \mid i\neq j]\, \Pr[i\neq j]$$

$$\geq E[\phi_i \mid i\neq j]\, \Pr[i\neq j]$$

$$\text{OptRev} = E[\phi_i] \leq 2\ \text{Rev. of Vickrey}.$$