SOCIAL WELFARE MAXIMIZATION

a.k.a. Economic efficiency.

- n agents ; m outcomes
- $V_i(j)$: value of agent i for outcome j.
- Goal : Find $j^* = \arg\max \sum_i V_i(j) =: SW_{j^*}$.

Combinatorial auctions : Outcomes are partitions of objects across agents.

Vickrey Clark Groves

VCG Mechanism

- Ask agents to report value functions.
- Compute $j^* = \arg\max \sum_i V_i(j)$
- Charge agents their "critical value".

$$p^{(i)} = \underbrace{\max_{j'} V_{-i}(j')}_{\text{Value of others optimized in absence of } i} - \underbrace{V_{-i}(j^*)}_{\text{Val. of others in presence of } i}$$

$p^{(i)}$: Independent of i's value once j^* is fixed

AGENT i's EXTERNALITY

Agent i's alternate view:

- Ask other agents to report values
- Compute

$$p_j^{(i)} = \max_{j'} V_{-i}(j') - V_{-i}(j)$$

CRITICAL PRICES

- Allow agent i to select outcome j^*.

Agent i will choose

$$\tilde{j}^* = \arg\max_j V_i(j) - p_j^{(i)}$$

Saw in prev. lec : $j^* = \tilde{j}^*$

Claim : VCG is Dominant Strategy Incentive Compatible

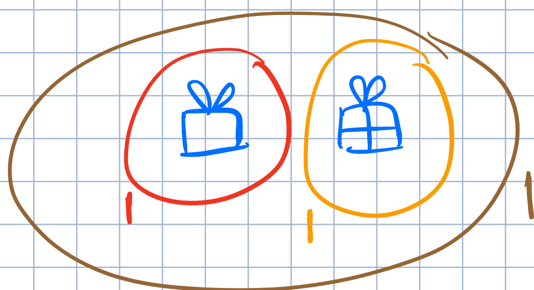
Corollary : VCG implements the efficient outcome.

Properties of VCG

- Efficient
- Dominant Strategy Incentive Compatible. (DSIC).

Problems with VCG.

- Needs exact computation of optimum — often computationally hard.
- Needs too much communication. — Spectrum Auctions
- Revenue non-monotonicity.
- Collusion ; false name bids.
- Credibility on part of the auctioneer.



Red & Brown agent
- Red wins & pays \$1.

R & B & O participate
- R & O win
- R pays 0
- O pays 0.

Special simpler case : Single-minded agents.

- n agents ; m items
- Agent i's value is given by (S_i, v_i)

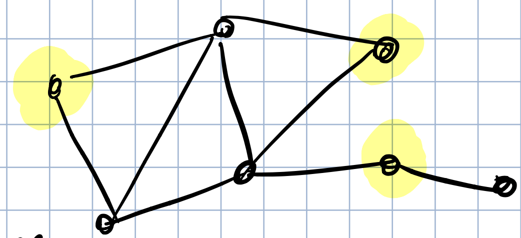
set of items i wants

value for that set

For $T \subseteq [m]$, $v_i(T) = \begin{cases} 0 & \text{if } T \not\supseteq S_i \\ v_i & \text{if } S_i \subseteq T \end{cases}$

Theorem: SW maximization is NP-Hard for single minded agents.

Independent Set
- NP-Hard to approximate better than $n^{1-\epsilon}$ for some $\epsilon > 0$



$n = \#$ vertices.

Reduction
- Agents are vertices.
- Items are edges. $m \approx n^2$.
- Each vertex will want all edges incident on it.
- Each vertex has value 1.

SW max. is hard to approx. better than $m^{1/2 - \epsilon}$ factor.

What about truthful approximations?

Possible "greedy" mechanism:

- Ask agents to report (S_i, v_i) $\quad(\tilde{S}_i, \tilde{v}_i)$
- Order agents in decreasing order of value $\quad$ value/$|S_i|$

  $v_1 \geq v_2 \geq \cdots \geq v_n.$ $\quad \frac{v_1}{|S_1|} \geq \frac{v_2}{|S_2|} \geq \frac{v_3}{|S_3|} \geq \cdots$
- $S = \emptyset$ $\quad$ (set of items allocated so far)
- For $i = 1$ to n,

  If $S_i \cap S = \emptyset$

  then allocate S_i to i and set $S = S \cup S_i$
- Charge i their critical price θ_i = min value at which i wins S_i.

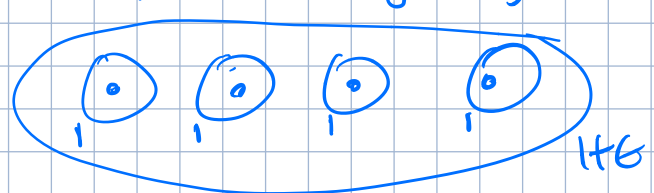
Red variation is DSIC.

Claim: Greedy alg. is DSIC. ✓

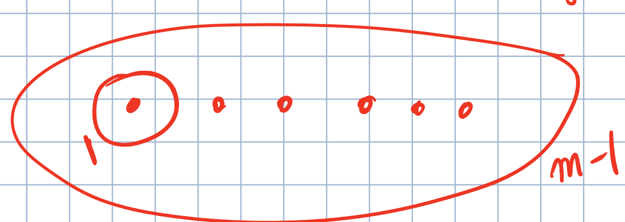
$\tilde{\theta}_i$ → i wins if $\tilde{v}_i \geq \tilde{\theta}_i$, util $= v_i - \tilde{\theta}_i$

loses if $\tilde{v}_i < \tilde{\theta}_i$, util $= 0$

Bad example: n agents; $n-1$ items.



Bad example for red variation: 2 agents, m items



The "LOS" mechanism.

Lehman O'Callaghan Shoham '02

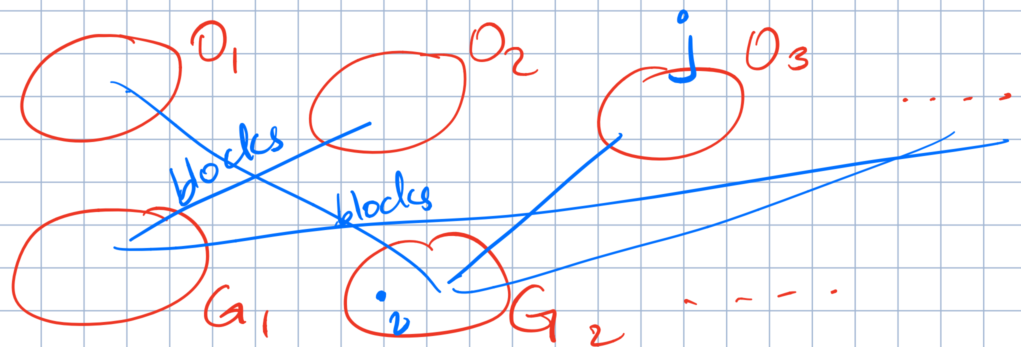
- Ask agents to report $(\tilde{S}_i, \tilde{v}_i)$
- Order agents in decreasing order of $\frac{\tilde{v}_i}{\sqrt{|S_i|}}$ $\quad f_i(\tilde{v}_i, \tilde{S}_i)$
- Greedily choose agents to serve in this order.
- Charge critical payments.

Claim: LOS is DSIC.

As long as $f_i(\tilde{v}_i, \tilde{S}_i)$ is non-decreasing in $\tilde{v}_i$.
and is non-increasing in $\tilde{S}_i$.
then the greedy mechanism is DSIC.

Claim: LOS achieves an $O(\sqrt{m})$-approximation.

Proof:



Every opt. set is blocked by some greedy set.
Consider i allocated by greedy. Let T_i = agents allocated in opt blocked by i.

$$\text{Let } j \in T_i \implies \frac{v_i}{\sqrt{|S_i|}} \geq \frac{v_j}{\sqrt{|S_j|}}$$

$$\implies v_j \leq v_i \frac{\sqrt{|S_j|}}{\sqrt{|S_i|}}$$

$\sqrt{|S_i|}$

Total optimal value blocked by i

$$= \sum_{j \in T_i} v_j \leq \frac{v_i}{\sqrt{|S_i|}} \sum_{j \in T_i} \sqrt{|S_j|}$$

$|T_i| \leq |S_i|$

$$\leq \frac{v_i}{\sqrt{|S_i|}} \sqrt{|T_i|} \sqrt{\sum_{j \in T_i} |S_j|}$$

$$\leq v_i \frac{\sqrt{|S_i|}}{\sqrt{|S_i|}} \cdot \sqrt{m}.$$

$$\leq \sqrt{m} \cdot v_i$$

$\Rightarrow$ OPT $\leq \sqrt{m}$ Value obtained by greedy.

A class of mechs. based on LOS

- Ask agents to report $(\tilde{S}_i, \tilde{v}_i)$
- Order agents in decreasing order of $f_i(v_i, S_i)$
- Greedily choose agents to serve in this order.
- Charge critical payments.

Claim: If $\forall i$, $f_i(v_i, S_i)$ is non-decreasing in v_i and non-increasing in S_i, then the above mechanism is DSIC.

What about other kinds of value functions?

A greedy algo by Borodin & Lucier '2011:

- Ask agents to report $\tilde{v}_i(\cdot)$.
- While items remain unallocated,
  - let $(i, X_i) = \operatorname{argmax}_{(i,S)\,:\,S \text{ is unallocated}} f_i(\tilde{v}_i, S)$
  - Allocate X_i to i.
- Charge each agent its critical price.

  θ_i = min reported value $\tilde{v}_i(X_i)$ at which i would win X_i.

No longer DSIC.

Assumption: Underlying greedy alg. is c-approx
f_i is non-dec. in $\tilde{v}_i$ and non-inc. in S

NASH EQUILIBRIUM

A tuple of strategies $(\sigma_1, \cdots, \sigma_n)$ forms a Nash Equil
if for all agents $i \in [n]$, fixing σ_{-i},
agent i's best response is to play σ_i.

Claim: Suppose that the functions f_i are monotone non-decreasing in v_i and the above greedy algorithm is a c-approximation.
Then, in any NE of the greedy mechanism, the SW obtained is a $(c+1)$-approx. to the optimal SW.

$$\boxed{PoA \leq c+1.}$$

1. Observation: bids $\leq$ values.

$$\sum_{\substack{(i,x_i) \text{ allocated} \\ \text{by greedy}}} v_i(x_i) \geq \sum_{\substack{(i,x_i) \text{ alloc.} \\ \text{by greedy}}} \tilde{v}_i(x_i)$$

2. $$\sum_{(i,x_i)} \tilde{v}_i(x_i) \geq \frac{1}{c} \sum_{(i,S_i) \in OPT} \theta_i(S_i)$$

3. Apply NE property.

$$\sum_{(i,S_i) \in OPT} v_i(S_i)$$

PRICE OF ANARCHY (maximization objective)

$$\max_{\text{Nash Equil.}} \frac{\text{Optimal obj.}}{\text{Objective at NE}}.$$

Ratio. (like approx ratio)

Worst case over all possible equilibria.