1. **(Integrality gaps.)** In both of the following cases try to find as large a gap as you can.

   (a) Give an integrality gap example for the facility location LP that we discussed in class. (That is, give an example for which the optimal LP solution is much smaller than the optimal integral solution.)

   (b) Given an integrality gap example for the Steiner tree LP that we discussed in class.

2. **(Integrality of the Min-cut polytope.)** Recall the following LP for the $s$-$t$ min-cut problem from class:

   $$\begin{align*}
   \min & \sum_{e \in E} c_e x_e \\
   \text{subject to} & \\
   \sum_{e \in P} x_e & \geq 1 \quad \forall s$-t$ paths $P$ in $G = (V, E)$ \\
   x_e & \geq 0 \quad \forall e \in E
   \end{align*}$$

   Prove that all basic solutions to this LP are integral.  
   (Hint: Show that any optimal fractional solution can be written as a convex combination of integral cuts.)

3. **(Cuts and $\ell_1$ metrics.)** A metric $d$ over a set $V$ is said to be an $\ell_1$ metric if the points can be mapped to points in $\mathbb{R}^k$ for some $k$, such that the distance between any two points according to $d$ is the $\ell_1$ distance between their mappings: $d(x, y) = \sum_i |x_i - y_i|$.  

   Also, a linear combination over cuts $\{\alpha_S\}_{S \subset V}$ defines the following metric $\mu_\alpha$ (verify that this is indeed a metric):

   $$\mu_\alpha(x, y) = \sum_{S \subset V, |S \cap \{x, y\}| = 1} \alpha_S \quad \forall x \neq y \in V$$

   (Note that $\sum_{S \subset V} \alpha_S$ is not necessarily equal to 1.)

   In this problem you will show that the above two classes of metrics—$\ell_1$ metrics and linear combinations of cuts—are in fact equivalent.

   (a) Prove that any metric defined by a linear combination of cuts is an $\ell_1$ metric.

   (b) Prove that any $\ell_1$ metric can be expressed as a linear combination of cuts.  
   (Hint: Prove this statement for a unit-dimensional $\ell_1$ metric first, that is, $k = 1$. Then extend it to multiple dimensions.)

4. **(Prize-collecting Steiner tree.)** The prize-collecting Steiner tree problem (PCST) is a variant of Steiner tree in which there are prizes $\pi_v$ on nodes and costs $c_e$ on edges, and a special node $r$ called the root. The goal is to construct a Steiner tree containing $r$ that minimizes the cost of the edges in the tree plus the value of the nodes not in the tree.

   (a) Give an LP relaxation for this problem using $x_e$ as an indicator of the extent to which an edge is included in the solution, and $y_v$ as an indicator of the extent to which a node is covered. (It is okay to have an exponential number of constraints, as for the Steiner forest LP we studied in class.)

   (b) Write the dual of the above LP.

   (c) Give a primal dual algorithm for this problem based on the one for Steiner tree (forest). (Don’t forget the pruning step!)
5. (Minimum-Cut linear arrangement.)

In the minimum-cut linear arrangement problem, we are given an unweighted graph $G = (V, E)$. Our goal is to find a one-to-one map from the $n$ vertices in $V$ to integers from 1 to $n$, such that the largest of the cuts $C_1, \cdots, C_{n-1}$ is minimized, where the cut $C_i$ is defined by the set of $i$ nodes mapped to integers 1 through $i$. For example, the picture below shows a linear arrangement with value 3.

Give a poly-log approximation for this problem.

(Hint: Use the Sparsest Cut or Balanced Cut algorithm from class.)