CS 880: Approximation Algorithms	**Homework 3**
Out: 3/15/07	**Due: 4/12/07**

1. **(Integrality gaps.)** In both of the following cases try to find as large a gap as you can.

   (a) Give an integrality gap example for the facility location LP that we discussed in class. (That is, give an example for which the optimal LP solution is much smaller than the optimal integral solution.)

   (b) Given an integrality gap example for the Steiner tree LP that we discussed in class.

2. **(Integrality of the Min-cut polytope.)** Recall the following LP for the s-t min-cut problem from class:

$$\min \sum_{e \in E} c_e x_e \qquad \text{subject to}$$
$$\sum_{e \in P} x_e \geq 1 \qquad \forall\ s\text{-}t \text{ paths } P \text{ in } G = (V, E)$$
$$x_e \geq 0 \qquad \forall\, e \in E$$

   Prove that all basic solutions to this LP are integral.

   (Hint: Show that any optimal fractional solution can be written as a convex combination of integral cuts.)

3. **(Cuts and ℓ_1 metrics.)** A metric $\mathbf{d}$ over a set V is said to be an ℓ_1 metric if the points can be mapped to points in $\mathbb{R}^k$ for some k, such that the distance between any two points according to $\mathbf{d}$ is the ℓ_1 distance between their mappings: $\mathbf{d}(x, y) = \sum_i |x_i - y_i|$.

   Also, a linear combination over cuts $\{\alpha_S\}_{S \subset V}$ defines the following metric μ_α (verify that this is indeed a metric):

$$\mu_\alpha(x, y) = \sum_{S \subset V : |S \cap \{x,y\}| = 1} \alpha_S \qquad \forall x \neq y \in V$$

   (Note that $\sum_{S \subset V} \alpha_S$ is not necessarily equal to 1.)

   In this problem you will show that the above two classes of metrics—ℓ_1 metrics and linear combinations of cuts—are in fact equivalent.

   (a) Prove that any metric defined by a linear combination of cuts is an ℓ_1 metric.

   (b) Prove that any ℓ_1 metric can be expressed as a linear combination of cuts. *(Hint: Prove this statement for a unit-dimensional ℓ_1 metric first, that is, $k = 1$. Then extend it to multiple dimensions.)*

4. **(Prize-collecting Steiner tree.)** The prize-collecting Steiner tree problem (PCST) is a variant of Steiner tree in which there are prizes π_v on nodes and costs c_e on edges, and a special node r called the root. The goal is to construct a Steiner tree containing r that minimizes the cost of the edges in the tree **plus** the value of the nodes **not** in the tree.

   (a) Give an LP relaxation for this problem using x_e as an indicator of the extent to which an edge is included in the solution, and y_v as an indicator of the extent to which a node is covered. (It is okay to have an exponential number of constraints, as for the Steiner forest LP we studied in class.)

   (b) Write the dual of the above LP.

   (c) Give a primal dual algorithm for this problem based on the one for Steiner tree (forest). (Don't forget the pruning step!)

(d) Prove that your algorithm achieves a 2-approximation. (If you cannot get the 2, try to get a slightly larger constant factor.)

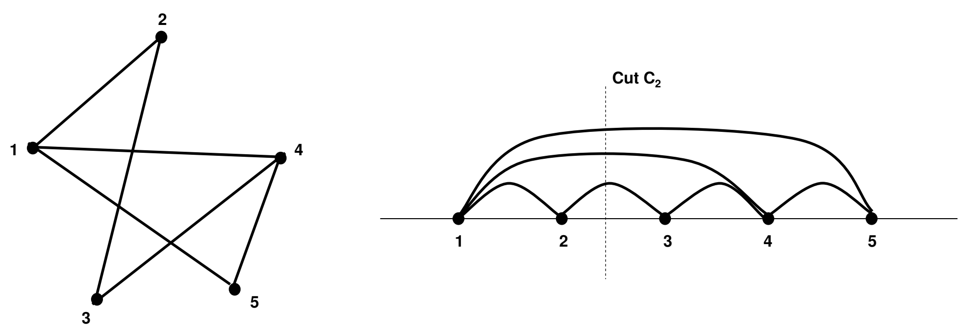
(Hint: Handle the component of the tree connected to the root separately from the other components. In each case, charge the cost of your solution to the value of the dual for that component.)

5. **(Minimum-Cut linear arrangement.)**

In the minimum-cut linear arrangement problem, we are given an unweighted graph $G = (V, E)$. Our goal is to find a one-to-one map from the n vertices in V to integers from 1 to n, such that the largest of the cuts $C_1, \cdots, C_{n-1}$ is minimized, where the cut C_i is defined by the set of i nodes mapped to integers 1 through i. For example, the picture below shows a linear arrangement with value 3.

Give a poly-log approximation for this problem.

(Hint: Use the Sparsest Cut or Balanced Cut algorithm from class.)



Graph

Linear Arrangement