CS880: Approximations Algorithms

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17.1 Facility Location

Recall from last time the formulation of the facility location problem as an LP with primal and dual. We have costs c_{ij} of routing a person j to facility i. There is a cost f_i of opening facility i. We have a variable x_i for each facility i that is the amount that the facility is open. We have a variable y_{ij} for the amount that person j is routed to facility i. We wish to minimize

$$\sum_i f_i x_i + \sum_{i,j} y_{ij} c_{ij}$$

suject to the constraints that for each j

$$y_{ij} \leq x_i$$

and for each j

$$\sum_i bij \geq 1$$

The dual problem then has variables α_j for every person j and β_{ij} for every customer/facility pair. The dual problem is to maximize

$$\sum_j \alpha_j$$

subject to the constraints that for any i, j

$$\alpha_j - \beta_{ij} \leq c_{ij}$$

and for any i

$$\sum_j \beta_{ij} \leq f_i$$

as well as the usual condition that all variables are non-negative.

17.2 Algorithm

We shall construct an integral solution to the primal LP as well as a feasible solution to the dual LP such that the integral primal is within a factor of three of the dual solution. Thus, our final solution shall be a three approximation for the facility location problem, since any dual feasible solution is an upper bound on a primal solution. As we usually do, we shall ensure that one of the complementary slackness conditions remains tight, while relaxing the other. This time, we shall relax the primal slackness condition. We shall do the following to get a set I of possible facilities to open. We continue until all the people are assigned a facility.

For all unassigned customers, raise the corresponding α_j uniformly. If an equaltiy of the form $\alpha_j = c_{ij}$ is reached the corresponding β_{ij} must be raised also to ensure that the constraint $\alpha_j - \beta_{ij} \leq c_{ij}$ isn't violated. Remember that we want this to be a feasible dual solution. For such customers, $\alpha_j - \beta_{ij}$ is fixed to be c_{ij} and that constraint is tight. Once a constraint of the form $\sum_j \beta_{ij} \leq f_i$ is reached for some i, we include this i in our set I of possible facilities to open. Also, we consider the j that have nonzero b_{ij} to be assigned to i and freeze the values of β_{ij} and α_i. Then we continue the process.

At this point, we have a set of facilities I that we wish to open. We call a pair i, j tight if $\alpha_j - \beta_{ij} = c_{ij}$ at the time i was included in the set I. We shall decide the final facilities to open by dong the following. There is a natural ordering on the facilities by the time at which they were included in I. We call a pair i, j tight if $\alpha_j = c_{ij}$. Now, we recursively do the following. Select the first facility in I that is not thrown out or already selected. Now, throw out all the facilities that share a tight customer with the selected facility. Continue until all facilities are selected or thrown out. The selected facilities shall be the ones we open, call this set of facilities S. We shall route each customer completely to a tight facility if one exists. If one doesn't exist, we route the customer to a facility that caused one of the tight facilities to the customer to be thrown out. By the metric propery, we know that the cost of routing to the open facility shall be bounded appropriately.

17.3 Analysis

For each customer, we break down the α_j into two portions, the portion payng for facility opening α_j^f and a portion for routing α_j^r. If j is assigned to a facility with which it is tight, then $\alpha_j = \beta_{ij} + c_{ij}$ for that facility. We call $\alpha_j^f = \beta_{ij}$ and $\alpha_j^r = c_{ij}$. Else, $\alpha_j^r = \alpha_j$ and the facility portion for this α_j is zero. Let S_i be the set of customers assigned to facility i. We notice that for any facility in S, $\sum_{j \in S_i} \alpha_j^f = \sum_{j \in S_i} \beta_{ij} = f_i$. Since these S_i are disjoint, $\sum_j \alpha_j^f$ is precisely the opening cost of the facilities in S. Now, for the routing costs we have that for j with a tight facility α_j^r is exactly the cost of routing j to that facility. For the other j, we don't have a tight facility to route to. But, the facility that j is routed to must share a customer, j', with a tight facility, i', for j. By the order in which we chose the facilities to include, the routing cost from i to j', from i' to j' and from i' to j are all bounded by α_j, since each of these pairs is tight. The metric property gives us that the cost paid to route j is then at most $3\alpha_j$. So, we have that:

$$3 \sum_j \alpha_j \geq \sum_{i \in S} f_i + \sum_{i,j} c_{ij}$$

Since the RHS is the objective function of the original problem and the optimal value for facility location is bounded by the sum of the α_j, we get a 3 approximation to facility location. This idea was first seen in [1].

17.4 Min Cut/Max Flow

Recall the Min Cut problem. We are given two nodes, s and t, in a graph G. We wish to find the cheapest set of edges to remove such that s and t are in different components of the graph. We shall consider the Min Cut problem in our primal dual setting. First, we must state the Min Cut problem as an ILP. To do this, we can give a variable x_e to each edge in the graph that stands for whether or not we remove e. Let c_e be the cost of removing e. We seek to minimize:

$$\sum_e c_e x_e$$

The constraints we have say that there should be no path from s to t that doesn't cross a selected edge. Thus, for every path P from s to t:

$$\sum_{e \in P} x_e \geq 1$$

As we saw in lecture 12, the dual of this LP describes the Max Flow problem. Using the Duality Theorem and the fact that the basic solutions of these LPs are integral, we get the famous Max Flow = Min Cut Theorem.

17.5 Metric LPs

Though the Min Cut problem is simple enough not to need another formulation, it is useful to use the Min Cut to expalin Metric LPs. Notice that any cut, C, defines a metric, d_C on the set of vertices. Namely, the distance $d_C(x, y)$ from x to y is length of the shortest path from x to y where the length traveled by moving across edge e is x_e. x_e is 1 if e is cut, zero otherwise. Now, our goal is to:

$$\begin{aligned} \text{Minimize} \quad & \sum_e c_e x_e && (17.5.1) \\ \text{Subject to} \quad & d_C(s,t) \geq 1 && (17.5.2) \end{aligned}$$

Unfortunately, the constraint $d_C(s, t) \geq 1$ is not linear in the variables x_e. To get over this, we change how we phrase the problem. Instead of assigning a varible x_e for each edge, we have a variable $d(u, v)$ for each pair of vertices. We shall want d to form a metric and shall think of $d(u, v)$ as being the amount we select the edge (if it exists) between u and v. Now we seek to minimize:

$$\sum_{(u,v) \in E} c_{(u,v)} d(u, v)$$

subject to the constraints that for any u, v, w:

$$d(u,v) + d(v,w) - d(u,w) \geq 0 \tag{17.5.3}$$

$$d(u,v) - d(v,u) = 0 \tag{17.5.4}$$

$$d(u,u) \geq 0 \tag{17.5.5}$$

$$d(s,t) \geq 1 \tag{17.5.6}$$

Notice that the first three of these constraints are perfectly encapsulates what it means for d to be a metric! Also, notice that the metric induced by any feasible cut satisfies the constraints. If the cut is optimal, then $\sum_{(u,v)\in E} c_{(u,v)} d(u,v) = \sum_e c_e x_e$ since we can lower the value of $x_{(u,v)}$ down to $d(u,v)$ without violating any of the contraints. Thus, finding the solution to Min Cut is equivalent to finding the best integral solution to the LP above.

As we will see in future lectures, phrasing some problems as a metric LP and using metric embedding techniques can lead to good approximation factors.

References

[1] K. Jain, V. Vazarani. Primal-Dual Algorithms for Mteric Facility Location and$k$-Median Problems. In *40th annual Symposium on the Foundations of Computer Science*, 1999.