1 The Vickrey Auction and Algorithmic Mechanism Design

1.1 Auctioning Off a Single Good

We begin by motivating combinatorial auctions, and the goals of algorithmic mechanism design more generally, with the following simple example. Suppose there is one item that we wish to sell to one of n candidate buyers, who we will also call *players* or *bidders*. The first basic assumption about bidders is the following.

(1) Each bidder i has a *valuation* v_i describing the bidder's "willingness to pay" for the item. This valuation is *private*, in the sense that the auctioneer and the other players have no information about it.

Informally, an *auction* is a protocol that interacts with the bidders, somehow determines a winner, and figures out a price p to charge the winner for the item. Our second basic assumption about bidders is the following.

(2) If bidder i loses, its utility is 0. If bidder i wins and has to pay the price p, its utility is the "residual worth" $v_i - p$.

The economic jargon for this assumption is that each bidder has *quasilinear utility*. While natural, one could argue with this assumption. For example, it ignores externalities (a loser doesn't care who the winner is); also, one could consider other utility functions that are increasing in the bidder's valuation and decreasing in the price. In this course, we will consider only quasilinear utility functions.

Periodically we pose questions like the following, inviting the reader to ponder an important point before reading further.

Question 1.1 How would you auction off the item? How would you argue that your auction is better than other ones?

To motivate our answer to this question, we first consider a protocol that all of you are familiar with. These days the first thing people think of when you say "auction" is probably eBay. Not that long ago, you might have thought about the climactic auction scene in many a movie and TV show. What happens in this auction? The auctioneer and the bidders are in the same room. The auctioneer names a price and the bidders willing to pay it raise their hands. The auctioneer raises the current price by small amounts, bidders successively drop out of the auction, and when only one bidder remains he/she wins at the current price. This is often called an *English (ascending) auction*.

To analyze this auction informally, first consider what behavior we expect from the bidders. As long as the current price p is less than bidder i's valuation v_i, we expect the bidder to stay in the auction—if it happens to win, its utility (recall (2)) will be strictly positive. On the other hand, if the current price p exceeds v_i then we expect the bidder to drop out—winning would now be undesirable, leading to negative utility. In summary, we expect each bidder to remain in the auction until the price hits its valuation, at which point we expect the bidder to drop out. We claim that it is now easy to figure out what outcome we expect.

Question 1.2 Which bidder wins in this ascending auction? What is the price paid?

Note that the winning bidder will be the one with the highest valuation (assume no ties for simplicity), and the price paid will be that when the second-to-last bidder drops out—when the price equals this bidder's valuation (possibly plus some small increment). In summary, we expect the bidder with the highest valuation to win and to pay the value of the *second-highest* valuation.

In 1961, Vickrey [11] had a very nice idea: if we know what outcome we expect from the English auction, why not do away with all the ceremony and just build it directly into an auction?

Precisely, the *Vickrey auction (VA)* is the following.

(1) Each bidder i submits a sealed bid b_i to an auctioneer (possibly $b_i = v_i$, possibly not).

(2) The auctioneer awards the item to the highest bidder.

(3) The auctioneer charges the winner a price equal to the second-highest bid.

Note that if every player sets $b_i = v_i$, then the Vickrey auction replicates the outcome of the English auction.

1.2 Good Properties of the Vickrey Auction

The Vickrey auction possesses a number of laudable qualities, which we now formalize. These six properties can also be regarded as a guiding list of desiderata for auctions in more complex settings (though in the interests of truth in advertising, we will never again be able to achieve all of them simultaneously). Because of this, some of the properties that we list are trivial for the VA and only become interesting for combinatorial auctions (see the next section).

The first property is the most important.

Proposition 1.3 *For every player i and for every set $\{b_j\}_{j \neq i}$ of bids for the other players, player i maximizes his/her utility by setting $b_i = v_i$. This holds even if player i knows the bids of the other players.*

Several comments are in order before we prove Proposition 1.3. Concisely, the proposition states that bidding truthfully (setting $b_i = v_i$) is never a bad idea. Because of this property, we say that the VA is *strategyproof* or *truthful*. In game theory parlance, bidding truthfully is a *dominant strategy*. In particular, player i need not care if the other players are bidding truthfully (cf., the Nash equilibrium concept). In Proposition 1.3, we are not assuming that player i knows all the other bids because we expect this to be the case (after all, it's a sealed-bid auction)—rather, awarding this clairvoyance to player i makes the truthfulness guarantee that much more compelling.

Several caveats. First, as mentioned earlier, we're assuming that all bidders possess a quasilinear utility function. Second, we are not claiming that truthtelling is always the *unique* way to maximize utility (though see Proposition 1.6 below). Third, we do not permit collusion by the players—we assume that a player i cannot influence the bids of the other players.

Proof of Proposition 1.3: Fix a player i with valuation v_i. Fix a bid b_j for each player $j \neq i$. We need to show that among all possible bids b_i for i, setting $b_i = v_i$ maximizes its utility. Let $B = \max_{j \neq i} b_j$ be the highest bid by one of the other players. Throughout this proof, for simplicity we ignore ties (the proof works with arbitrary tie breaking, as you should check). There are then two cases.

First suppose that $v_i < B$. Note that if the player bids truthfully it will lose and obtain zero utility. This remains true if the player bids anything less than B. If the player bids more than B, however, it will win and pay B. Its utility is then $v_i - B < 0$, lower than it is when i bids truthfully.

Now suppose that $v_i > B$. If i bid truthfully, i wins and enjoys positive utility $v_i - B > 0$. If the player bids below B then it loses, receives zero utility, and is worse off than before. The final case is really the key point of the Vickrey auction: *no matter what i bids above B, its price (B) and hence its utility ($v_i - B$) remain the same.*

In both cases, there is no false bid that yields strictly higher utility than a truthful one, so the proof is complete. ■

As a point of contrast, note that Proposition 1.3 (and in particular the final case in the proof) fails for a *first-price auction*—the auction obtained by replacing the third step of the VA with charging the highest bidder its own bid. (If bidder i knew B and $v_i > B$, then it would bid $B + \epsilon$ for small ϵ.)

Question 1.4 Ponder some other variations of the VA (e.g., third-price auctions) and whether or not they are strategyproof.

Question 1.5 Where would collusion disrupt the proof of Proposition 1.3?

The proof of Proposition 1.3 shows that when player i knows the bids of the other players, there are many different bids that maximize utility (of which truthtelling is always one). On the other hand, when the player does *not* know the other bids, then every false bid can come back to haunt the player.

Proposition 1.6 *For every bid $b_i \neq v_i$, there is a set of bids $\{b_j\}_{j \neq i}$ by the other players such that i's utility would have been strictly larger had it bid truthfully ($b_i = v_i$).*

Proof: If $b_i < v_i$, choose the other bids so that B (the highest bid) satisfies $b_i < B < v_i$ (so that i loses instead of winning and getting positive utility). If $b_i > v_i$, choose bids so that $b_i > B > v_i$ (so that i wins and incurs negative utility instead of losing). ■

Sometimes you hear auctions satisfying both Proposition 1.3 and 1.6 called *strongly truthful*, and those satisfying only Proposition 1.3 *weakly truthful*. We will typically not give Proposition 1.6 much thought, though most (if not all) of the auctions that we discuss satisfy an analogous guarantee.

The final four propositions are trivial, and we single them out only because of their relevance for more general combinatorial auctions. The first states that the utility of truthtellers is always nonnegative in the VA.

Proposition 1.7 *Truthtelling bidders always receive nonnegative utility in the VA.*

Proof: Losers receive zero utility. The price charged to the winner is at most its bid; if its bid equals its valuation, then the resulting utility is nonnegative. ■

More economic jargon: auctions satisfying Proposition 1.7 are called *individually rational (IR)*, or are said to have the *voluntary participation (VP)* property.

We call Propositions 1.3, 1.6, and 1.7 *incentive constraints*, in that they are all meant to ensure that bidders behave in a predictable, desirable way: bidding their true valuations.

At this point you might well ask: why is truthtelling important? We give two reasons, one from the perspective of the participants, and the other from the perspective of the auctioneer.

(1) In a first-price auction, knowledge about other bidders can be useful in determining what to bid. Bidders are therefore motivated to expend resources to gain such information. Of course, all such bidders are similarly motivated, potentially making the outcome of the auction hard to predict. In a truthful auction, knowledge about other bids is irrelevant; every bidder is justified (subject to the caveats mentioned earlier) in sitting back, relaxing, and just bidding their valuation.

(2) When bidders report their true valuations, the auctioneer is in position to solve an underlying optimization problem that involves the private valuations. (Without truthful bids, there is essentially no way to solve such an optimization problem.)

To elaborate on the second point, consider the objective function of maximizing the *social surplus*, defined as

$$\max \sum_{i=1}^{n} v_i x_i, \tag{1}$$

where x_i is 1 (0) if i is a winner (loser). Obviously, we impose the constraint that $\sum_i x_i = 1$. This is sometimes called the *utilitarian* objective function.

For a single-item auction, maximizing the social surplus simply means giving the item to the player who values it the most. Note that optimizing this objective function intuitively requires precise knowledge about the highest (private) valuation.

Remark 1.8 The price is not part of the surplus—in this context, prices are viewed as a "transfer" between players and the auctioneer that permits implementation of the socially best outcome, rather than as a loss in social welfare. We can also view the auctioneer as a non-strategic player whose utility function is the revenue that it earns; this utility then cancels out the utility lost by the auction winner from paying for the item. We emphasize, however, that in these notes we do not view the auctioneer as a player that actively seeks to maximize its revenue. See the companion course notes by Jason Hartline for auctions that are designed to maximize the auctioneer's revenue.

Since the VA awards the item to the highest bidder, it is an *(economically) efficient* auction in the sense that it maximizes the surplus.

Proposition 1.9 *If all players bid truthfully, then the VA produces an outcome that maximizes the social surplus.*

Note that the incentive constraints (Propositions 1.3, 1.6, and 1.7) are meant to ensure the hypothesis in Proposition 1.9.

Next we note that the VA makes no assumption about valuations. For example, we do not assume that valuations are bounded above by a known constant. (Strictly speaking, we are assuming that the value of losing is 0. We also usually think of the value of winning as being nonnegative. The VA does not essentially depend on either of these assumptions, however.)

Proposition 1.10 *The VA works with general valuations.*

Finally, a focal point in our discussion of combinatorial auctions: the VA is computationally tractable, in that it can be implemented in polynomial (indeed, linear) time.

Proposition 1.11 *The VA is a polynomial-time auction.*

1.3 Summary

This section introduced the Vickrey auction and formalized the pseudo-theorem that it is "a good auction". Specifically, we identified four desirable properties of the VA.

(P1) It satisfies strong incentive constraints (Proposition 1.3, 1.6, and 1.7).

(P2) It is economically efficient, in the sense that it maximizes the surplus (1) (Proposition 1.9).

(P3) It works with general valuations (Proposition 1.10).

(P4) It is a poly-time auction (Proposition 1.11).

We will see that these properties are not simultaneously achievable in the richer domain of combinatorial auctions, and will seek to understand the feasible trade-offs between them.