Packing multiway cuts in capacitated graphs

Siddharth Barman

Computer Sciences Dept., University of Wisconsin-Madison.

Joint work with Shuchi Chawla

Outline

Multiway Cut Packing

- Given: A graph G and k commodities, each corresponding to a set of vertices.
- Goal: Produce a collection of cuts $\{C_1, C_2, .., C_k\}$, such that C_i is a multiway cut for commodity i
- Objective: Maximum "load" on any edge is minimized.

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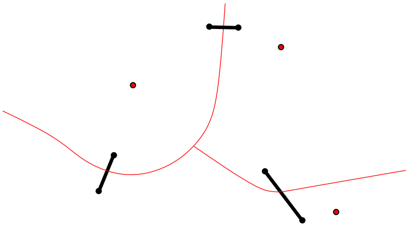

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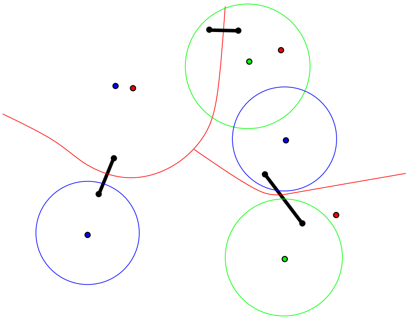

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- Mutiway cut packing belongs to the class of graph labeling problem, where the goal is to complete a given partial labeling in the most consistent possible way.

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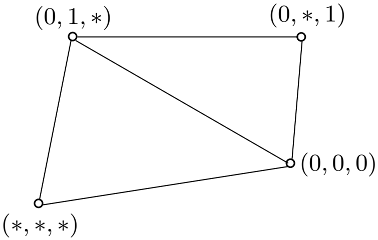

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- Mutiway cut packing is *NP*-complete.

Related work

- Rabani, Schulman and Swamy [SODA'08], introduced the multicut packing problem, generalizing the taxonomical labeling problem.

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- Using LP relaxation and randomized rounding they obtained an $O\left(\frac{\log n}{\log \log n}\right)$ approximation
- They developed a deterministic $O(\log^2 k)$ approximation algorithm for trees.

Results

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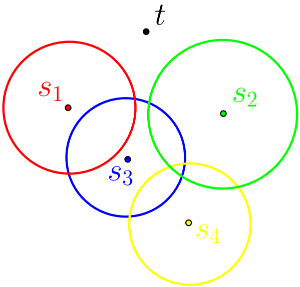
- For the common sink case we provide an algorithm that guarantees a maximum load of $OPT + 2$.



Approach

$\boxed{\text{LP Relaxation}}$

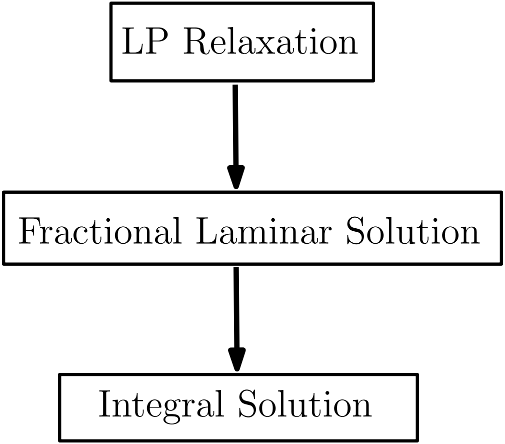
Approach

LP Relaxation

$\downarrow$

Fractional Laminar Solution

Approach


LP Relaxation
Fractional Laminar Solution
Integral Solution

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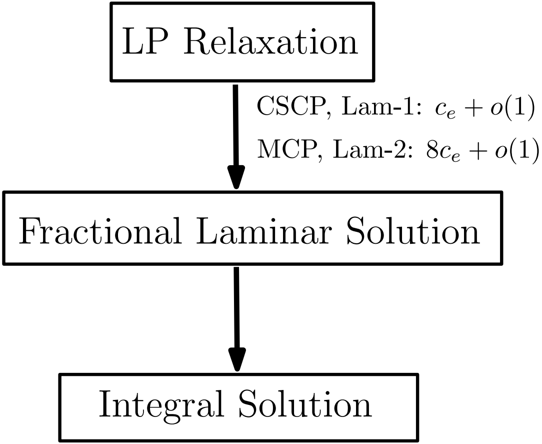
LP Relaxation
CSCP, Lam-1: $c_e + o(1)$
MCP, Lam-2: $8c_e + o(1)$
Fractional Laminar Solution
Integral Solution

Approach

LP Relaxation

↓ CSCP, Lam-1: $c_e + o(1)$
MCP, Lam-2: $8c_e + o(1)$

Fractional Laminar Solution

↓ CSCP, Round-1: $c_e + 1$
MCP, Round-2: $c_e + 3$

Integral Solution

Multiway Cut Packing: Integer Program and LP relaxation

$$\begin{aligned} \sum_{e \in P} x_{a,e} &\geq 1 && \forall a \in [k], P \in \mathcal{P}_a \\ \sum_{a} x_{a,e} &\leq c_e && \forall e \in E \\ x_{a,e} &\in \{0,1\} && \forall a \in [k], e \in E \end{aligned}$$

$$\begin{aligned} d_a(u,v) &\leq d_a(u,w) + d_a(w,v) && \forall a \in [k], u, v, w \in V \\ d_a(r_i, r_j) &\geq 1 && \forall a \in [k], i, j \in S_a \\ \textstyle\sum_a d_a(e) &\leq c_e && \forall e \in E \\ d_a(e) &\in [0,1] && \forall a \in [k], e \in E \end{aligned}$$

Definitions

load $l_e^{\mathcal{C}}$ Recall that given a collection of cuts $\mathcal{C} = \{C_1, C_2, .., C_k\}$ the load $l_e^{\mathcal{C}}$ on edge e is the number of cuts that e crosses, that is $l_e^{\mathcal{C}} = |\{C_i \ : \ e \in \delta(C_i), C_i \in \mathcal{C}\}|$

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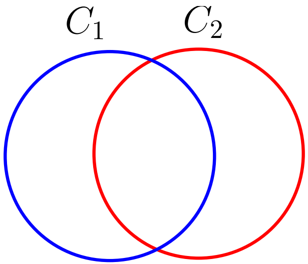
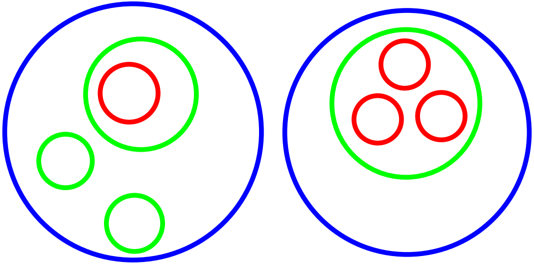
A pair of cuts C_1, C_2 are said to *cross* if all of the sets $C_1 \cap C_2, C_1 \setminus C_2, C_2 \setminus C_1$ are non-empty.



Definitions

Laminarity A collection of cuts $\mathcal{C} = \{C_1, C_2, ..., C_k\}$ is said to be laminar if no pair of cuts $C_i, C_j \in \mathcal{C}$ cross



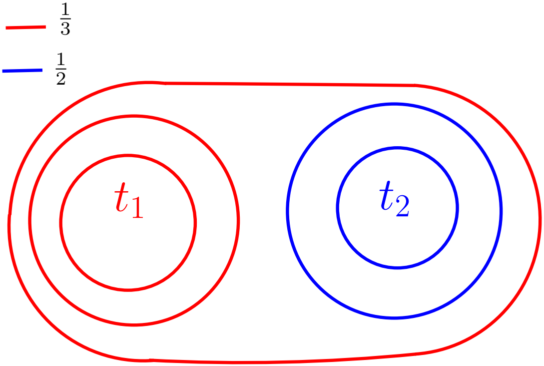
Definitions

A fractional laminar cut family $\mathcal{C}$ for terminal set T with weight function w is a collection of cuts with the following properties:

- The collection is laminar
- Each cut C in the family is associated with a unique terminal in T. We use $\mathcal{C}_i$ to denote the sub-collection of sets associated with terminal $i \in T$. Every $C \in \mathcal{C}_i$ contains the node r_i.
- For all $i \in T$, the total weight of cuts in $\mathcal{C}_i$, $\sum_{C \in \mathcal{C}_i} w(C)$, is 1.

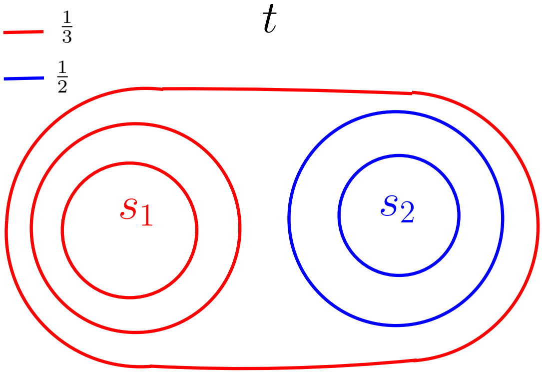
Definitions

Fractional laminar cut family $\mathcal{C}$ for $T = \{t_1, t_2\}$



Definitions

Feasible fractional laminar cut family $\mathcal{C}$ for CSCP with terminal set $T = \{s_1, s_2\}$



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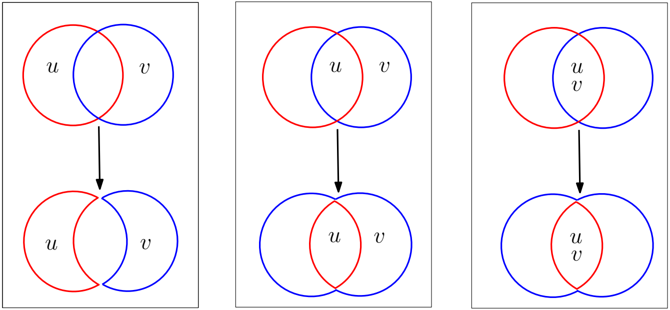

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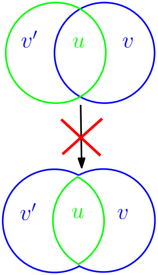
- The multiway case is more involved than the common sink case primarily because pair wise separation of terminals needs to be maintained.



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- We start simple transformations as before when applicable and then applying a conflict graph resolution transformation constructed over terminals whose cuts cross.
- We essentially reassign subsets of original cuts in a particular order whilst ensuring that the loads do not go beyond a multiplicative factor of 2.

Rounding fractional cut family: CSCP

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- $l_e^{\mathcal{C}}$ denotes the load of the fractional cut family
- $l_e^{\mathcal{A}}$ denotes the load of the integral cut family

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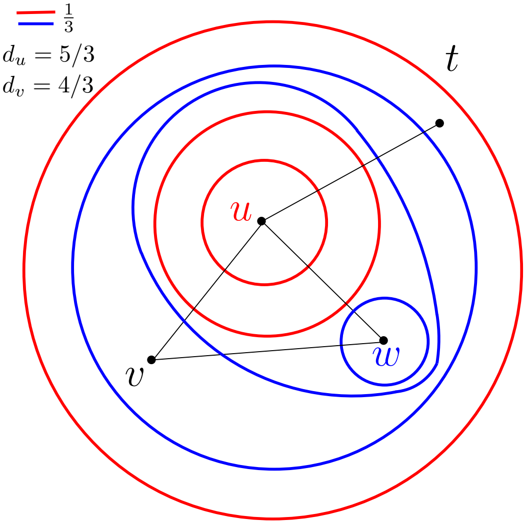

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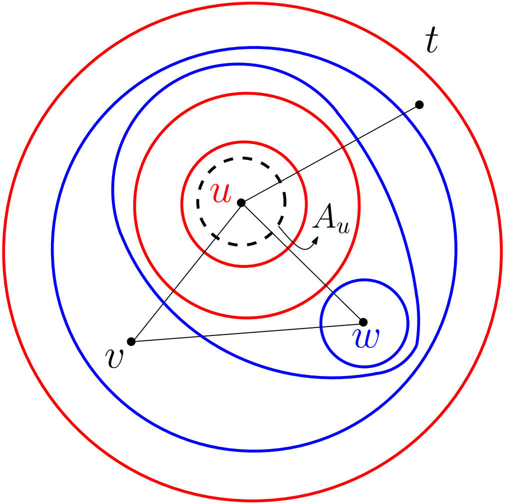

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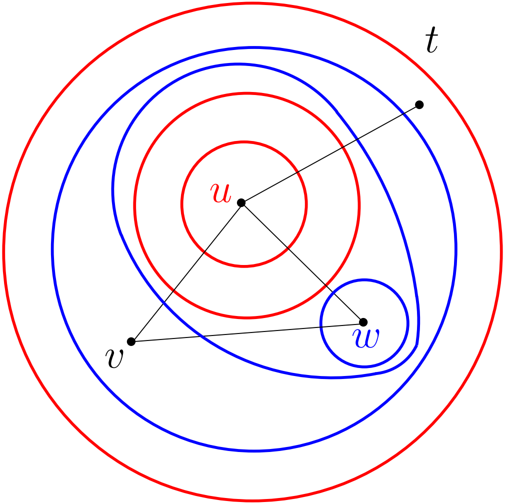

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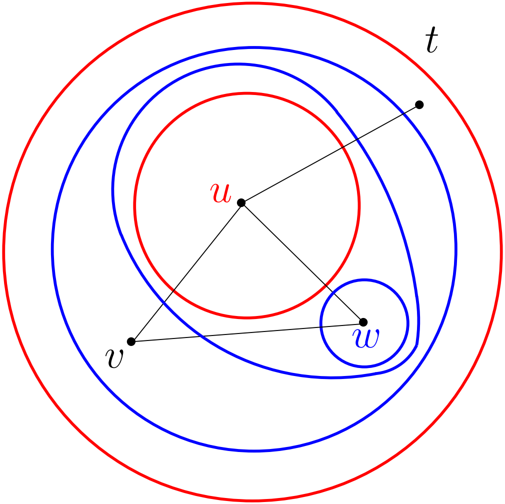

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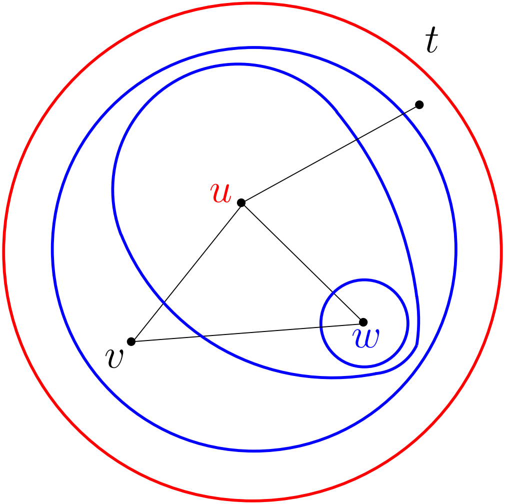

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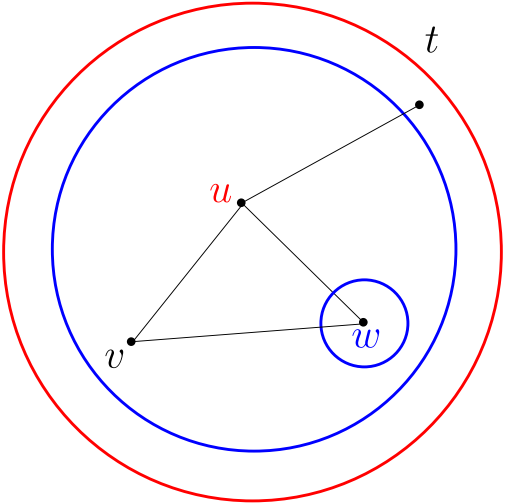

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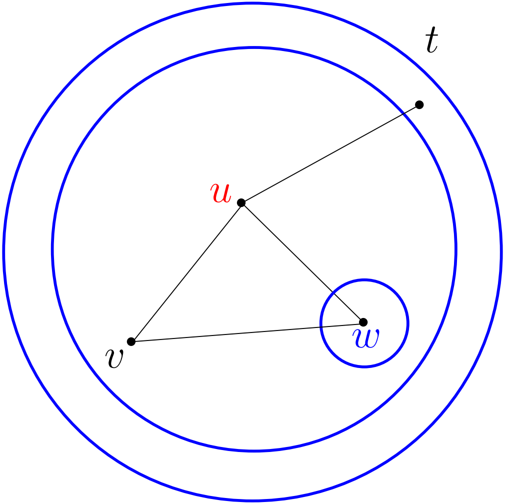

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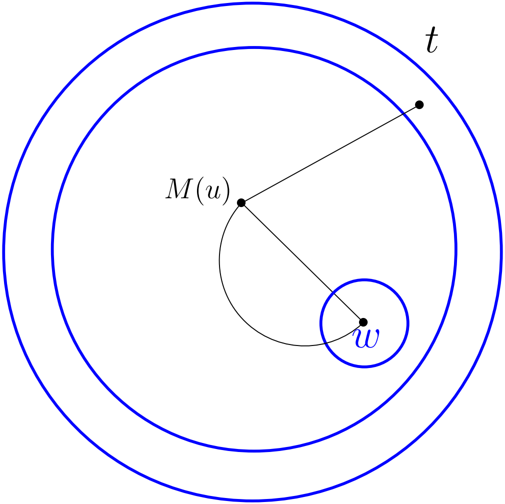

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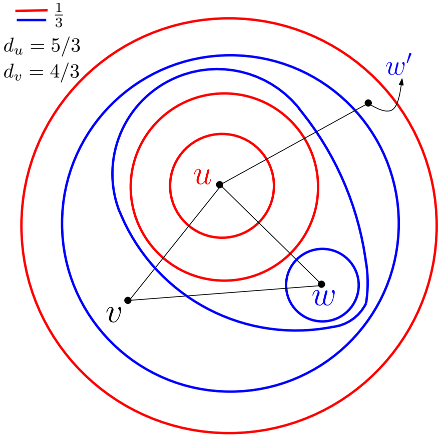
Cost analysis

The integral load on an edge $l_e^{\mathcal{A}}$ never goes above $c_e + 1$

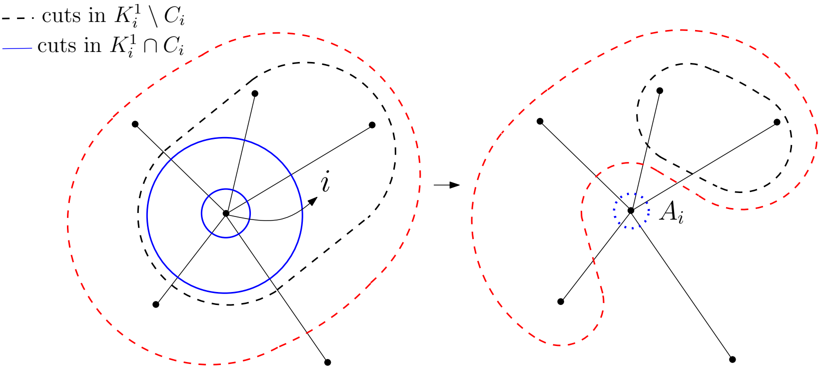
Rounding fractional cut family: MCP

Unlike CSCP in MCP simple reassignment does not suffice to maintain separation



Rounding fractional cut family: MCP

Hence we shift fractional cuts in order to maintain a feasible fractional laminar cut family for the unassigned terminals: $C \leftarrow C \setminus \{i\}$



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- In MCP the fractional load on edges goes up but the algorithm ensures that this overload is not beyond 3
- Hence given a laminar feasible fractional cut family with edge capacities c_e we can obtain integral cuts with edge capacities $c_e + 3$

Rounding fractional cut family: MCP

- All in all, given a feasible instance of the MCP with edge capacities c_e the Laminarity Algorithm followed by Rounding algorithm produces a family of multicuts such that load on any edge is no more than $8c_e + 4$

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- All in all, given a feasible instance of the MCP with edge capacities c_e the Laminarity Algorithm followed by Rounding algorithm produces a family of multicuts such that load on any edge is no more than $8c_e + 4$
- Similarly for CSCP the Laminarity Algorithm followed by the Rounding algorithm produces integral cuts with load no more than $c_e + 2$

Open Problems

- Set Multiway Packing and Multicut Packing

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- Is the “laminarity gap” small for Multicut Packing problem as well?

Thank you!