Packing multiway cuts in capacitated graphs

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Joint work with Shuchi Chawla
Multiway Cut Packing

- Given: A graph $G$ and $k$ commodities, each corresponding to a set of vertices.
- Goal: Produce a collection of cuts \{ $C_1, C_2, \ldots, C_k$ \}, such that $C_i$ is a multiway cut for commodity $i$
- Objective: Maximum “load” on any edge is minimized.
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Motivation

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- Mutiway cut packing is $NP$-complete.
Rabani, Schulman and Swamy [SODA’08], introduced the multicut packing problem, generalizing the taxonomical labeling problem.
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Related work

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- They developed a deterministic $O(\log^2 k)$ approximation algorithm for trees.
We present an LP-based approximation algorithm for multiway cut packing problem in general graphs that guarantees a maximum edge load of at most $8OPT + 4$. For the common sink case we provide an algorithm that guarantees a maximum load of $OPT + 2$. 
Results

- We present an LP-based approximation algorithm for multiway cut packing problem in general graphs that guarantees a maximum edge load of at most $8\OPT + 4$.
- For the common sink case we provide an algorithm that guarantees a maximum load of $\OPT + 2$. 

![Diagram of sets S1, S2, S3, S4 intersecting at a point t]
Approach

LP Relaxation
Approach

LP Relaxation → Fractional Laminar Solution
Approach

LP Relaxation

Fractional Laminar Solution

Integral Solution
Approach

LP Relaxation

CSCP, Lam-1: \( c_e + o(1) \)
MCP, Lam-2: \( 8c_e + o(1) \)

Fractional Laminar Solution

Integral Solution
**Approach**

- **LP Relaxation**
  - CSCP, Lam-1: \( c_e + o(1) \)
  - MCP, Lam-2: \( 8c_e + o(1) \)

- **Fractional Laminar Solution**
  - CSCP, Round-1: \( c_e + 1 \)
  - MCP, Round-2: \( c_e + 3 \)

- **Integral Solution**
Multiway Cut Packing: Integer Program and LP relaxation

\[
\sum_{e \in P} x_{a,e} \geq 1 \quad \forall a \in [k], P \in \mathcal{P}_a
\]
\[
\sum_a x_{a,e} \leq c_e \quad \forall e \in E
\]
\[
x_{a,e} \in \{0, 1\} \quad \forall a \in [k], e \in E
\]

\[
d_a(u, v) \leq d_a(u, w) + d_a(w, v) \quad \forall a \in [k], u, v, w \in V
\]
\[
d_a(r_i, r_j) \geq 1 \quad \forall a \in [k], i, j \in S_a
\]
\[
\sum_a d_a(e) \leq c_e \quad \forall e \in E
\]
\[
d_a(e) \in [0, 1] \quad \forall a \in [k], e \in E
\]
Definitions

\textbf{load} \ l^C_e \quad \text{Recall that given a collection of cuts} \quad \mathcal{C} = \{C_1, C_2, \ldots, C_k\} \quad \text{the load} \ l^C_e \quad \text{on edge} \ e \quad \text{is the number of cuts that} \ e \quad \text{crosses, that is} \quad l^C_e = |\{C_i : e \in \delta(C_i), C_i \in \mathcal{C}\}|
Recall that given a collection of cuts $\mathcal{C} = \{C_1, C_2, .., C_k\}$ the load $l^C_e$ on edge $e$ is the number of cuts that $e$ crosses, that is

$$l^C_e = |\{C_i : e \in \delta(C_i), C_i \in \mathcal{C}\}|$$

A pair of cuts $C_1, C_2$ are said to cross if all of the sets $C_1 \cap C_2, C_1 \setminus C_2, C_2 \setminus C_1$ are non-empty.
Definitions

**Laminarity** A collection of cuts $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ is said to be laminar if no pair of cuts $C_i, C_j \in \mathcal{C}$ cross.
Definitions

A fractional laminar cut family $\mathcal{C}$ for terminal set $T$ with weight function $w$ is a collection of cuts with the following properties:

- The collection is laminar
- Each cut $C$ in the family is associated with a unique terminal in $T$. We use $C_i$ to denote the sub-collection of sets associated with terminal $i \in T$. Every $C \in C_i$ contains the node $r_i$.
- For all $i \in T$, the total weight of cuts in $C_i$, $\sum_{C \in C_i} w(C)$, is 1.
Definitions

Fractional laminar cut family $\mathcal{C}$ for $T = \{t_1, t_2\}$
Definitions

Feasible fractional laminar cut family \( C \) for CSCP with terminal set \( T = \{ s_1, s_2 \} \)
Obtaining fractional laminar cut family: CSCP

- LP solution for CSCP $\rightarrow$ feasible fractional non-laminar family of cuts
Obtaining fractional laminar cut family: CSCP

- LP solution for CSCP → feasible fractional *non-laminar* family of cuts
- For CSCP apply following rules for transforming an arbitrary cut family into a laminar one which is feasible over edge capacities $c_e + o(1)$. 
Obtaining fractional laminar cut family: CSCP

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- For CSCP apply following rules for transforming an arbitrary cut family into a laminar one which is feasible over edge capacities $c_e + o(1)$.
Obtaining fractional laminar cut family: MCP

- LP solution for MCP → Feasible fractional laminar cut family $\mathcal{C}$ that is feasible for the MCP with edge capacities $8c_e + o(1)$
Obtaining fractional laminar cut family: MCP

- LP solution for MCP → Feasible fractional laminar cut family $\mathcal{C}$ that is feasible for the MCP with edge capacities $8c_e + o(1)$
- The multiway case is more involved than the common sink case primarily because pair wise separation of terminals needs to be maintained.
Obtaining fractional laminar cut family: MCP

- We use a more complex set of transformation rules to convert an integral family of multiway cuts into a laminar one.
Obtaining fractional laminar cut family: MCP

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- We start simple transformations as before when applicable and then applying a conflict graph resolution transformation constructed over terminals whose cuts cross.
Obtaining fractional laminar cut family: MCP

- We use a more complex set of transformation rules to convert an integral family of multiway cuts into a laminar one.
- We start simple transformations as before when applicable and then applying a conflict graph resolution transformation constructed over terminals whose cuts cross.
- We essentially reassign subsets of original cuts in a particular order whilst ensuring that the loads do not go beyond a multiplicative factor of 2.
Rounding fractional cut family: CSCP

\[ C \rightarrow A \]
Rounding fractional cut family: CSCP

\[ \mathcal{C} \rightarrow \mathcal{A} \]

\( K_v \) denotes the set of cuts in \( \mathcal{C} \) that contain \( v \)
Rounding fractional cut family: CSCP

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\( K_v \) denotes the set of cuts in \( C \) that contain \( v \)

\( d_v \) is the total weight of all cuts in \( K_v \):

\[ d_v = \sum_{C \in K_v} w(C) \]
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- \( l^C_e \) denotes the load of the fractional cut family
Rounding fractional cut family: CSCP

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\( l^C_e \) denotes the load of the fractional cut family

\( l^A_e \) denotes the load of the integral cut family
Rounding fractional cut family: CSCP

\[ d_u = \frac{5}{3} \]
\[ d_v = \frac{4}{3} \]
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Correctness
Throughout the algorithm, the cut family $C$ is a feasible fractional laminar family for the unassigned terminals.
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Throughout the algorithm, the cut family $\mathcal{C}$ is a feasible fractional laminar family for the unassigned terminals.

Cost analysis

The integral load on an edge $l_e^A$ never goes above $c_e + 1$
Rounding fractional cut family: MCP

Unlike CSCP in MCP simple reassignment does not suffice to maintain separation

\[ d_u = \frac{5}{3} \]
\[ d_v = \frac{4}{3} \]
Hence we **shift** fractional cuts in order to maintain a feasible fractional laminar cut family for the unassigned terminals: 
\[ C \leftarrow C \setminus \{i\} \]
In MCP the fractional load on edges goes up but the algorithm ensures that this overload is not beyond 3.
Rounding fractional cut family: MCP

- In MCP the fractional load on edges goes up but the algorithm ensures that this overload is not beyond 3.
- Hence given a laminar feasible fractional cut family with edge capacities $c_e$ we can obtain integral cuts with edge capacities $c_e + 3$. 
Rounding fractional cut family: MCP

All in all, given a feasible instance of the MCP with edge capacities $c_e$ the Laminarity Algorithm followed by Rounding algorithm produces a family of multicuts such that load on any edge is no more than $8c_e + 4$
Rounding fractional cut family: MCP

- All in all, given a feasible instance of the MCP with edge capacities $c_e$ the Laminarity Algorithm followed by Rounding algorithm produces a family of multicuts such that load on any edge is no more than $8c_e + 4$

- Similarly for CSCP the Laminarity Algorithm followed by the Rounding algorithm produces integral cuts with load no more than $c_e + 2$
Open Problems

- Set Multiway Packing and Multicut Packing
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- Set Multiway Packing and Multicut Packing
- Is the “laminarity gap” small for Multicut Packing problem as well?
Thank you!