The Complexity of Nash Equilibria as Revealed by Data

Siddharth Barman

Joint work with Umang Bhaskar, Federico Echenique, and Adam Wierman
Model

\[
\begin{pmatrix}
2 & 7 & 
\vdots & 1 \\
8 & 2 & 
\ddots & 8 \\
\vdots & \ddots & \ddots & \ddots \\
18 & 28 & \cdots & 4 \\
\end{pmatrix}, \quad \begin{pmatrix}
3 & 1 & 
\vdots & 4 \\
1 & 5 & 
\ddots & 9 \\
\vdots & \ddots & \ddots & \ddots \\
26 & 5 & \cdots & 35 \\
\end{pmatrix}
\]

E.g. Normal Form Game

Algorithm

Solution

Prob.

Player 1

Prob.

Player 2

E.g. Nash Equilibrium
Computing equilibria in economic models, in general, is computationally hard.

For example, Nash equilibrium in two-player normal form (bimatrix) game is PPAD-hard [DGP06, CDT09].
Model

\[
\begin{pmatrix}
2 & 7 & \cdots & 1 \\
8 & 2 & \cdots & 8 \\
\vdots & \vdots & \ddots & \vdots \\
18 & 28 & \cdots & 4
\end{pmatrix},
\begin{pmatrix}
3 & 1 & \cdots & 4 \\
1 & 5 & \cdots & 9 \\
\vdots & \vdots & \ddots & \vdots \\
26 & 5 & \cdots & 35
\end{pmatrix}
\]

E.g. Normal Form Game

Solution

Algorithm

Hard even in bimatrix games [DGP06, CDT09]

E.g. Nash Equilibrium

Player 1

Prob.

Player 2

Prob.
<table>
<thead>
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<td>E.g. Nash Equilibrium</td>
</tr>
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</table>

\[
\begin{pmatrix}
2 & 7 & \cdots & 18 & 2 \\
8 & 2 & \cdots & 8 & 3 \\
2 & 1 & \cdots & 2 & 1 \\
18 & 28 & \cdots & 4 & 26 \\
\end{pmatrix}
\times
\begin{pmatrix}
3 & 1 & \cdots & 9 \\
1 & 5 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
26 & 5 & \cdots & 35 \\
\end{pmatrix}
\]

- **Algorithm**
  - Hard even in bimatrix games [DGP06, CDT09]

- **Solution**
  - Prob. Player 1
  - Prob. Player 2
Observed Behavior

Model

\[
\begin{pmatrix}
2 & 7 & \cdots & 9 \\
3 & 2 & \cdots & 8 \\
\vdots & \vdots & \ddots & \vdots \\
18 & 28 & \cdots & 4
\end{pmatrix}
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
2 & 5 & \cdots & 4 \\
\vdots & \vdots & \ddots & \vdots \\
26 & 5 & \cdots & 3
\end{pmatrix}
\]

E.g. Normal Form Game

Algorithm

Hard even in bimatrix games

Solution

[DGP06, CDT09]

Observed Behavior

E.g. Nash Equilibrium

Prob.

Player 1

Prob.

Player 2

E.g. Nash Equilibirum
Observed Behavior

Model

Solution

Revealed Preference

E.g. Normal Form Game

Algorithm

Hard even in bimatrix games
[DGP06, CDT09]

Observed Behavior

Revealed Preference

E.g. Normal Form Game

Algorithm

Hard even in bimatrix games
[DGP06, CDT09]

Observed Behavior

Revealed Preference

E.g. Normal Form Game

Algorithm

Hard even in bimatrix games
[DGP06, CDT09]

Observed Behavior

Revealed Preference

E.g. Normal Form Game

Algorithm

Hard even in bimatrix games
[DGP06, CDT09]
For given observed behavior,

- Does there exist a rationalization?
- Does there exist a rationalization with a particular form?

Fundamental questions in revealed preference theory (adopted from Varian’s survey [2006]).
For given observed behavior,

- Does there exist a **rationalization**?
- Does there exist a **rationalization with a particular form**?

Prior work for

- Consumer Choice Theory [Samuelson '38,...]
- Pure Nash Equilibrium [Sprumont '00,...]
- ...
For given observed behavior,

- Does there exist a rationalization?
- Does there exist a rationalization with a particular form?

Our Focus: Nash Equilibria in Bimatrix Games
For given observed behavior,

- Does there exist a **rationalization**?
- Does there exist a **computationally-tractable rationalization**?

**Motivation:** Among rationalizations of similar explanatory power a tractable one provides a more credible rationale.
Bimatrix Games

Payoff matrices $A$, $B$ of size $n \times n$

With players’ mixed strategies $x \in \Delta_n$ and $y \in \Delta_n$

Row player’s payoff: $x^T Ay$

Column player’s payoff: $x^T By$
Bimatrix Games

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Best response sets

$$\text{BR}_r(y) := \{ i \mid e_i^T Ay \geq e_k^T Ay \ \forall k \}$$
$$\text{BR}_c(x) := \{ j \mid x^T Be_j \geq x^T Be_k \ \forall k \}$$
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Nash equilibrium $(x, y)$:

$$\text{Supp}(x) \subseteq \text{BR}_r(y) \quad \text{and}$$
$$\text{Supp}(y) \subseteq \text{BR}_c(x)$$
Bimatrix Games

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$$\text{BR}_r(y) := \{ i \mid e_i^T Ay \geq e_k^T Ay \quad \forall k \}$$

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Strict Nash equilibrium $(x, y)$:

$$\text{Supp}(x) = \text{BR}_r(y) \quad \text{and} \quad \text{Supp}(y) = \text{BR}_c(x)$$
Data set $D = \{ (x_k, y_k) \in \Delta_n \times \Delta_n \}_k$
Data set $D = \{ (x_k, y_k) \in \Delta_n \times \Delta_n \}_k$

**Definition (Rationalizable)**

A data set $D = \{(x_k, y_k)\}_k$ is said to **rationalizable** iff there exists payoff matrices $A$ and $B$ such that $(x_k, y_k)$ is a **strict** Nash equilibrium in $(A, B)$, for all $k$. 
Data set $D = \{ (x_k, y_k) \in \Delta_n \times \Delta_n \}_k$

**Definition (Rationalizable)**

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**Proposition**

We can determine in polynomial time whether a data set has a rationalization.

\[
\begin{align*}
\text{maximize} & \quad \delta \\
\text{subject to} & \quad (Ay_k)_i = \pi_k, \quad \forall k, \forall i \in \text{Supp}(x_k) \\
& \quad (Ay_k)_j \leq \pi_k - \delta, \quad \forall k, \forall j \notin \text{Supp}(x_k) \\
& \quad (x_k^T B)_i = \pi'_k, \quad \forall k, \forall i \in \text{Supp}(y_k) \\
& \quad (x_k^T B)_j \leq \pi'_k - \delta, \quad \forall k, \forall j \notin \text{Supp}(y_k) \\
& \quad 0 \leq A_{i,j}, B_{i,j} \leq 1 \quad \forall i, j \in [n] \\
& \quad \delta \geq 0.
\end{align*}
\]
For given observed behavior:

✓ Does there exist a rationalization?

• Does there exist a computationally-tractable rationalization?
Classes of games that admit efficient computation of Nash equilibrium:

- Zero-sum games
- Potential games
- $\text{rank}(A + B) \leq 1$ [AGMS11]
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But, the above mentioned classes can be ruled out by constant-size data sets.
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**Definition (Player Rank)**

The player rank of a game \((A, B)\) equals

\[
\min\{\text{rank}(A), \text{rank}(B)\}
\]

Intuition: If rank of \(A\) is \(k\) then the row player has \(k\) representative strategies.
Classes of games that admit efficient computation of Nash equilibrium:

- Zero-sum games
- Potential games
- \( \text{rank}(A + B) \leq 1 \) [AGMS11]

But, the above mentioned classes can be ruled out by constant-size data sets.

**Definition (Player Rank)**

The player rank of a game \((A, B)\) equals

\[
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\]

**Theorem [GJM11]**

If player rank of a game is \( k \) then equilibria can be computed in time \( O(n^{O(k)}) \)
For given observed behavior:

✓ Does there exist a **rationalization**?

• Does there exist a **computationally-tractable rationalization**?
For given observed behavior:

✓ Does there exist a *rationalization*?

• Does there exist a *low player rank rationalization*?
Our Results (Informally)

Player rank of rationalization is (almost) minimum of:

- **Dimensionality** of observed mixed strategies.
- **Support size** of the observed mixed strategies.
- **Chromatic number** of the data set.

Structurally Simple Data

\[\downarrow\]

Low Player Rank Rationalization

\[\downarrow\]

Efficient Computation of Nash Eq.
Data Set $D = \{(x_k, y_k)\}_k$
Data Set $D = \{(x_k, y_k)\}_{k}$

$\dim d$

$\text{rank}(B) \leq d$
Data Set $D = \{(x_k, y_k)\}_k$

Theorem

If a data set $D = \{(x_k, y_k)\}_k$ is rationalizable then it is rationalizable by a game of player rank at most $\min\{\dim(\{x_k\}_k), \dim(\{y_k\}_k)\}$.
Data Set $D = \{(x_k, y_k)\}_k$

**Theorem**

If a data set $D = \{(x_k, y_k)\}_k$ is rationalizable then it is rationalizable by a game of player rank at most $\min\{\dim(\{x_k\}_k), \dim(\{y_k\}_k)\}$.

**Corollary**

If a data set $D = \{(x_k, y_k)\}_k$ is rationalizable then it is rationalizable by a game of player rank at most $|D|$. 
Support Size

Data set $D = \{(x_k, y_k)\}_{k}$
Support Size

Data set \( D = \{(x_k, y_k)\}_k \)

If \( |\text{Supp}(x_k)| \leq s \) for all \( k \) then \( \text{rank}(A) \leq 2s + 1 \)
Data set $D = \{(x_k, y_k)\}_k$

If $|\text{Supp}(x_k)| \leq s$ for all $k$ then $\text{rank}(A) \leq 2s + 1$

If $|\text{Supp}(y_k)| \leq s$ for all $k$ then $\text{rank}(B) \leq 2s + 1$
Theorem

If data set $D = \{(x_k, y_k)\}_k$ is generic and $|\text{Supp}(x_k)| \leq s$ for all $k$ or $|\text{Supp}(y_k)| \leq s$ for all $k$ then $D$ can be rationalized by a game with player rank at most $2s + 1$. 
Data set $D = \{(x_k, y_k)\}_{k}$

If $|\text{Supp}(x_k)| \leq s$ for all $k$ and $\{y_k\}_k$ linearly independent then

$\text{rank}(A) \leq 2s + 1$
Data set $D = \{(x_k, y_k)\}_k$

If $|\text{Supp}(x_k)| \leq s$ for all $k$ and $\{y_k\}_k$ linearly independent then $\text{rank}(A) \leq 2s + 1$

Fact:

Polynomials $p_1(x), p_2(x), \ldots, p_m(x)$ each of degree $d$
Data set \( D = \{(x_k, y_k)\}_k \)

If \(|\text{Supp}(x_k)| \leq s\) for all \(k\) and \(\{y_k\}_k\) linearly independent then \(\text{rank}(A) \leq 2s + 1\)

Fact:

Polynomials \( p_1(x), p_2(x), \ldots, p_m(x) \) each of degree \(d\)

\[
M = \begin{pmatrix}
p_1(1) & p_2(1) & \cdots & p_m(1) \\
p_1(2) & p_2(2) & \cdots & p_m(2) \\
\vdots & \vdots & \ddots & \vdots \\
p_1(n) & p_2(n) & \cdots & p_m(n)
\end{pmatrix}
\]

\(n \times m\) matrix \(M\)
Data set $D = \{(x_k, y_k)\}_{k}$

If $|\text{Supp}(x_k)| \leq s$ for all $k$ and $\{y_k\}_k$ linearly independent then $\text{rank}(A) \leq 2s + 1$

Fact:

Polynomials $p_1(x), p_2(x), \ldots, p_m(x)$ each of degree $d$

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\vdots & \vdots & \ddots & \vdots \\
p_1(n) & p_2(n) & \cdots & p_m(n)
\end{pmatrix}$$

$\text{rank}(M) \leq d + 1$
Data set \( D = \{(x_k, y_k)\}_k \)

If \(|\text{Supp}(x_k)| \leq s\) for all \( k \) and \( \{y_k\}_k \) linearly independent then \( \text{rank}(A) \leq 2s + 1 \)

**Proof:** For \((x_k, y_k)\) want matrix \( A \) such that \( \text{BR}_r(y_k) = \text{Supp}(x_k) \)
Data set $D = \{(x_k, y_k)\}_k$

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**Proof:** For $(x_k, y_k)$ want matrix $A$ such that $BR_r(y_k) = \text{Supp}(x_k)$

Polynomial $p_k(x) := -\prod_{i\in\text{Supp}(x_k)}(x - i)^2$
Data set \( D = \{(x_k, y_k)\}_k \)

If \(|\text{Supp}(x_k)| \leq s\) for all \(k\) and \(\{y_k\}_k\) linearly independent then \(\text{rank}(A) \leq 2s + 1\)

**Proof:** For \((x_k, y_k)\) want matrix \(A\) such that \(BR_r(y_k) = \text{Supp}(x_k)\)

Polynomial \(p_k(x) := -\prod_{i \in \text{Supp}(x_k)} (x - i)^2\)

\(\text{Supp}(x_k) = \{2, 5\}\)
Data set \( D = \{(x_k, y_k)\}_k \)

If \( |\text{Supp}(x_k)| \leq s \) for all \( k \) and \( \{y_k\}_k \) linearly independent then \( \text{rank}(A) \leq 2s + 1 \)

**Proof:** For \((x_k, y_k)\) want matrix \( A \) such that \( \text{BR}_r(y_k) = \text{Supp}(x_k) \)

Polynomial \( p_k(x) := -\prod_{i \in \text{Supp}(x_k)}(x - i)^2 \)
Data set \( D = \{(x_k, y_k)\}_k \)

If \(|\text{Supp}(x_k)| \leq s\) for all \(k\) and \(\{y_k\}_k\) linearly independent then \(\text{rank}(A) \leq 2s + 1\)

Proof: For \((x_k, y_k)\) want matrix \(A\) such that \(\text{BR}_r(y_k) = \text{Supp}(x_k)\)

\[
\begin{pmatrix}
1 \\
2 \\
\vdots \\
n
\end{pmatrix}
\begin{pmatrix}
\text{max components}
\end{pmatrix}
\begin{pmatrix}
\text{max components}
\end{pmatrix}
\begin{pmatrix}
< 0 \\
0 \\
< 0 \\
\vdots \\
0 \\
< 0
\end{pmatrix}
\]

Polynomial \(p_k(x) := -\prod_{i \in \text{Supp}(x_k)} (x - i)^2\)
Data set $D = \{(x_k, y_k)\}_k$

If $|\text{Supp}(x_k)| \leq s$ for all $k$ and $\{y_k\}_k$ linearly independent then

$$\text{rank}(A) \leq 2s + 1$$

Proof: For $(x_k, y_k)$ want matrix $A$ such that $BR_r(y_k) = \text{Supp}(x_k)$

Polynomial $p_k(x) := -\prod_{i \in \text{Supp}(x_k)} (x - i)^2$
Data set $D = \{(x_k, y_k)\}_k$

If $|\text{Supp}(x_k)| \leq s$ for all $k$ and $\{y_k\}_k$ linearly independent then

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**Proof:** For $(x_k, y_k)$ want matrix $A$ such that $\text{BR}_r(y_k) = \text{Supp}(x_k)$

Polynomial $p_k(x) := -\prod_{i \in \text{Supp}(x_k)}(x - i)^2$

$$P := \begin{pmatrix} p_1(1) & \cdots & p_k(1) & \cdots & p_m(1) \\ p_1(2) & \cdots & p_k(2) & \cdots & p_m(2) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ p_1(m) & \cdots & p_k(n) & \cdots & p_m(n) \end{pmatrix}$$
Data set \( D = \{(x_k, y_k)\}_k \)

If \( |\text{Supp}(x_k)| \leq s \) for all \( k \) and \( \{y_k\}_k \) linearly independent then \( \text{rank}(A) \leq 2s + 1 \)

**Proof:** For \((x_k, y_k)\) want matrix \( A \) such that \( \text{BR}_r(y_k) = \text{Supp}(x_k) \)

Polynomial \( p_k(x) := -\prod_{i \in \text{Supp}(x_k)} (x - i)^2 \)

\[
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p_1(1) & \cdots & p_k(1) & \cdots & p_m(1) \\
p_1(2) & \cdots & p_k(2) & \cdots & p_m(2) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
p_1(m) & \cdots & p_k(n) & \cdots & p_m(n)
\end{pmatrix}
\]

\[\arg \max_i P_{i,k} = \text{Supp}(x_k)\]
Data set \( D = \{(x_k, y_k)\}_k \)

If \( |\text{Supp}(x_k)| \leq s \) for all \( k \) and \( \{y_k\}_k \) linearly independent then \( \text{rank}(A) \leq 2s + 1 \)

**Proof:** For \((x_k, y_k)\) want matrix \( A \) such that \( BR_r(y_k) = \text{Supp}(x_k) \)

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\end{pmatrix}
\]

Set \( A = PV \) where \( Vy_k = e_k \)
Data set $D = \{(x_k, y_k)\}_k$

If $|\text{Supp}(x_k)| \leq s$ for all $k$ and $\{y_k\}_k$ linearly independent then $\text{rank}(A) \leq 2s + 1$

**Proof:** For $(x_k, y_k)$ want matrix $A$ such that $\text{BR}_r(y_k) = \text{Supp}(x_k)$

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  \vdots & \ddots & \vdots & \ddots & \vdots \\
p_1(m) & \cdots & p_k(n) & \cdots & p_m(n)
\end{pmatrix}$$

Set $A = PV$ where $Vy_k = e_k$

$Ay_k = Pe_k \Rightarrow \text{BR}_r(y_k) = \text{Supp}(x_k)$
Data set $D = \{(x_k, y_k)\}_k$

If $|\text{Supp}(x_k)| \leq s$ for all $k$ and $\{y_k\}_k$ linearly independent then \[\text{rank}(A) \leq 2s + 1\]

**Proof:** For $(x_k, y_k)$ want matrix $A$ such that $\text{BR}_r(y_k) = \text{Supp}(x_k)$

Polynomial $p_k(x) := - \prod_{i \in \text{Supp}(x_k)} (x - i)^2$

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Set $A = PV$ where $Vy_k = e_k$

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p_1(m) & \cdots & p_k(n) & \cdots & p_m(n)
\end{pmatrix} $$

Set $A = PV$ where $Vy_k = e_k$

$\text{rank}(A) \leq \text{rank}(P) \leq 2s + 1$
✓ **Dimensionality**: if either player has at most $d$ linearly independent mixed strategies the there is a rationalization of player rank $d$.

✓ **Support size**: if either player has mixed strategies of support size at most $s$ then there is a rationalization of player rank $2s + 1$.

• **Chromatic number** of the data set.
Row chromatic number $\kappa_R$ equals chromatic number of graph that represents row intersections.
Row chromatic number $\kappa_R$ equals chromatic number of graph that represents row intersections.

Column chromatic number $\kappa_C$ equals chromatic number of graph that represents column intersections.
Row chromatic number $\kappa_r$ equals chromatic number of graph that represents row intersections.

Column chromatic number $\kappa_c$ equals chromatic number of graph that represents column intersections.
Row chromatic number $\kappa_r$ equals chromatic number of graph that represents row intersections.

Column chromatic number $\kappa_c$ equals chromatic number of graph that represents column intersections.

**Theorem**

If data set $D$ is **generic** then $D$ can be rationalized by a game with player rank at most $2 \min\{\kappa_r, \kappa_c\}$. 
Our Results (Informally)

✓ Dimensionality = $d \Rightarrow$ Player rank $\leq d$

✓ Support size = $s \Rightarrow$ Player rank $\leq 2s + 1$

✓ Chromatic number = $\kappa \Rightarrow$ Player rank $\leq 2\kappa$
Our Results (Informally)

✓ Dimensionality = $d$ ⇒ Player rank $\leq d$

✓ Support size = $s$ ⇒ Player rank $\leq 2s + 1$

✓ Chromatic number = $\kappa$ ⇒ Player rank $\leq 2\kappa$

Can unify these results:
If generic $D = D_1 \cup D_2 \cup D_3$ such that
- $\dim(D_1) \leq d$
- Supp. size in $D_2 \leq s$
- Chromatic number of $D_3 \leq \kappa$

then player rank $\leq 2(d + s + \kappa) + 1$. 
Our Results (Informally)

✓ Dimensionality $= d \Rightarrow$ Player rank $\leq d$

✓ Support size $= s \Rightarrow$ Player rank $\leq 2s + 1$

✓ Chromatic number $= \kappa \Rightarrow$ Player rank $\leq 2\kappa$

**Lower bounds:**
There exists data $D$ set such that any rationalization of $D$ has

- Player rank $\geq d$
- Player rank $\geq s$
- Player rank $\geq \kappa$
Our Results (Informally)

✓ Dimensionality = $d \Rightarrow$ Player rank $\leq d$

✓ Support size = $s \Rightarrow$ Player rank $\leq 2s + 1$

✓ Chromatic number = $\kappa \Rightarrow$ Player rank $\leq 2\kappa$

Matching lower bounds:
There exists data $D$ set such that any rationalization of $D$ has

• Player rank $\geq d$
• Player rank $\geq s$
• Player rank $\geq \kappa$

Thank you!