Region Growing for Multi-route Cuts

Siddharth Barman

Joint work with Shuchi Chawla

Outline

- Multi-route Cuts and LP Formulation
- Region growing lemma
- Algorithms

Connectivity

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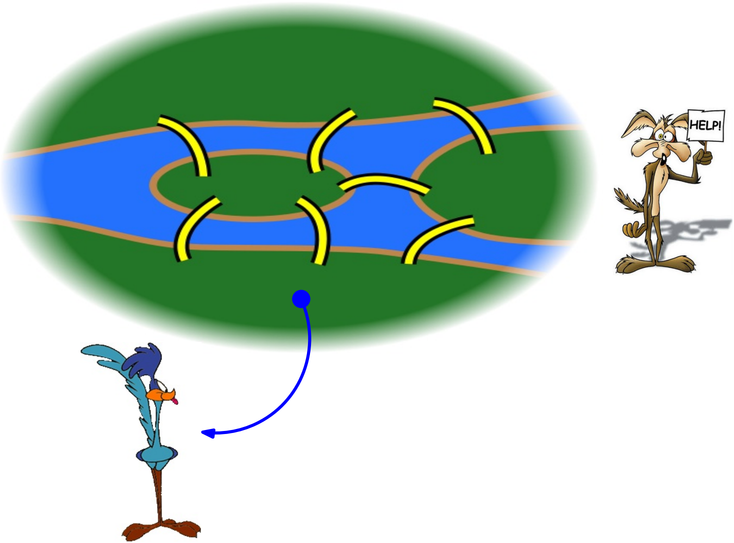
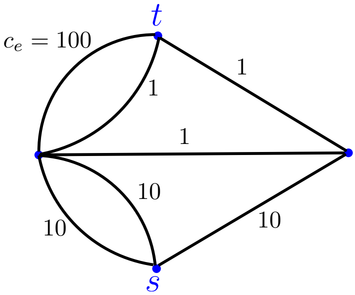
- In classical cut problems the goal is to find a set of edges whose removal reduces the connectivity to zero.
- Relaxing to higher connectivity requirements gives us multi-route cuts.

Road Runner in Königsberg



Multi-route Cut Instance

- Objective: Find a low cost set of edges such that connectivity between s to t to reduces to 1.



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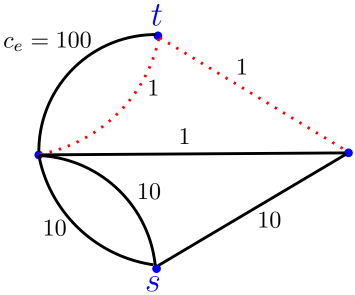

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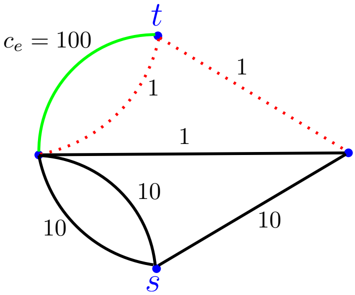

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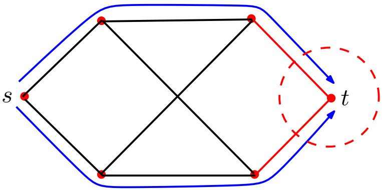
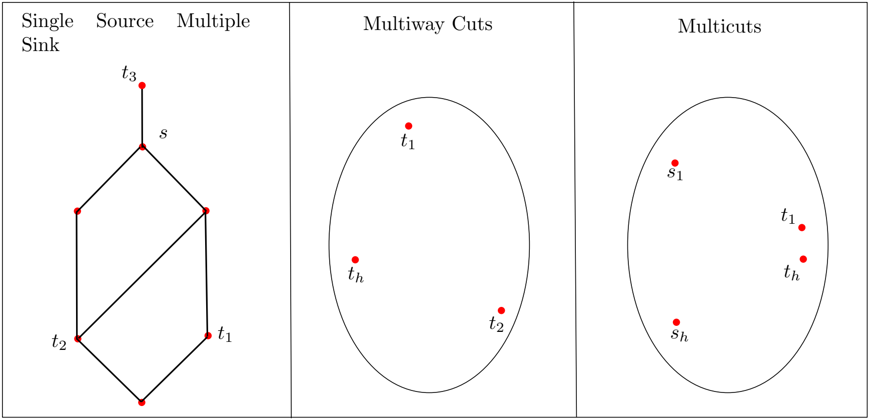
- Connectivity := Number of edge disjoint paths = 2.
- Menger's Theorem: ∃ cut C with exactly 2 edges.



Spectrum of Cut Problems



Multicut version of multi-route cut

- Given: A graph G with edge costs with terminal pairs, $\{(s_1, t_1), (s_2, t_2), \cdots, (s_h, t_h)\}$, and connectivity threshold, k.
- Goal: To produce a minimum cost set of edges $E' \subseteq E$, such that for each i, s_i and t_i are at most $(k-1)$-edge-connected in the graph $(V, E \setminus E')$.

Related Work

- Bruhn, Cerný, Hall and Kolman [SODA07] gave a duality theorem for multi-route flows and cuts on uniform capacity graphs.
- Chekuri and Khanna [ICALP08], gave a randomized $O(\log^2 n \ \log h)$ approximation algorithm for 2-route multicuts.
- Classical Multicut via region growing by Garg, Vazirani and Yannakakis [STOC03]

Results

- EDRC: Edge Disjoint Route Cut
- NDRC: Node Disjoint Route Cut
- SS: Single Source Multiple Sink, MW: Multiway Cut, MC: Multicut,

Problem	Previous best result	Our result
SS-2-EDRC, SS-2-NDRC	$O(\log n)$	$O(\log h)$
MW-2-EDRC, MW-2-NDRC	$O(\log n \log h)$	$O(\log^2 h)$
MC-2-EDRC	$O(\log^2 n \log h)$	$O(\log^2 h)$
MC-2-NDRC	–	$O(\log^2 h)$
SS-k-EDRC	–	$(6, O(\sqrt{h} \ln h))$
SS-k-EDRC-Uniform	–	$(2, 4)$
SS-k-EDRC (constant h)	–	$(4, 4)$

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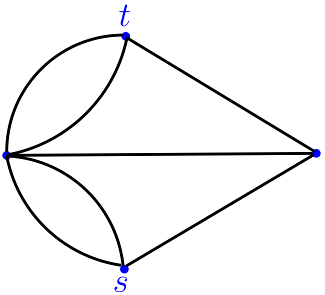
LP Formulation for Multicut Version

$$
\begin{aligned}
\tilde{z} = \quad & \min \sum_{e \in E} x_e \; c_e \\
\text{s.t} \quad & \sum_{e \in E} y_e^i \quad \leq k - 1 && \forall i \in [h] \\
& d^i(u,v) \quad = \; x_e + y_e^i && \forall i \in [h], e = (u,v) \in E \\
& d^i \text{ is a metric} && \forall i \in [h] \\
& d^i(s_i, t_i) \quad \geq 1 && \forall i \in [h]
\end{aligned}
$$

Guessing Witness Edges

- We can guess $k-1$ witness edges and then find min-cut.
- But does not scale up for:
    - Higher k values, as there are $\binom{n}{k-1}$ possibilities
    - Multiple terminals



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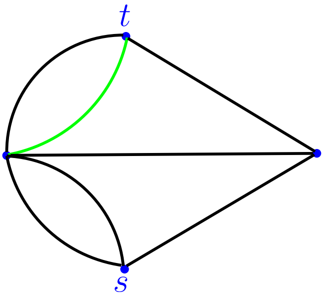

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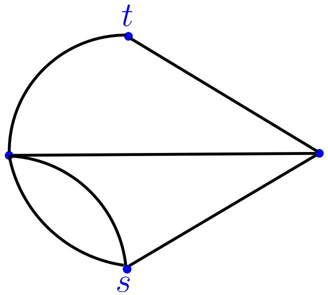

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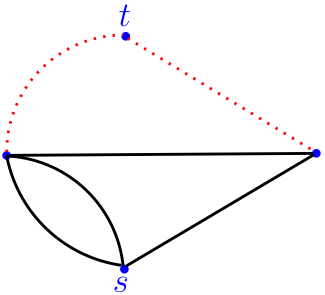
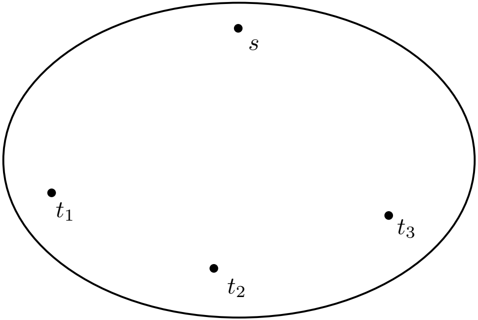


Construction Highlight: $k = 2$

- Let $cost_1(C)$ be the total cost of the cut minus the most expensive edge
- Say, for s and all t_i, there exists a cut C such that $cost_1(C)$ $\leq \alpha \times$ LP contribution contained inside C.



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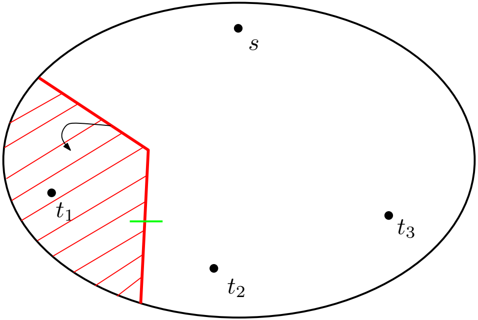

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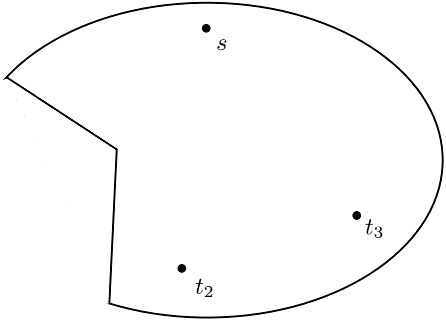

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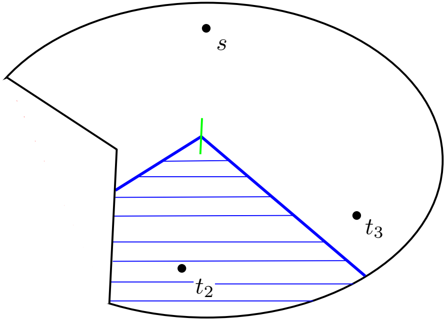

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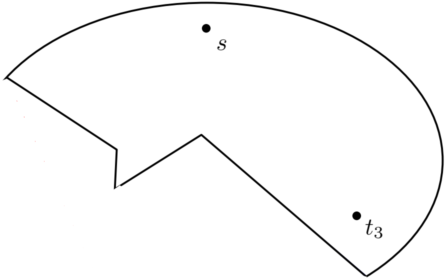

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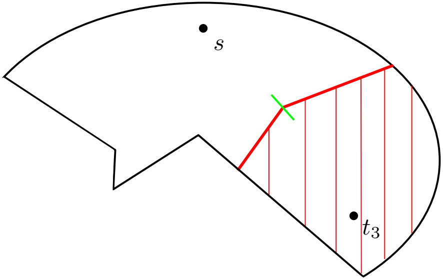

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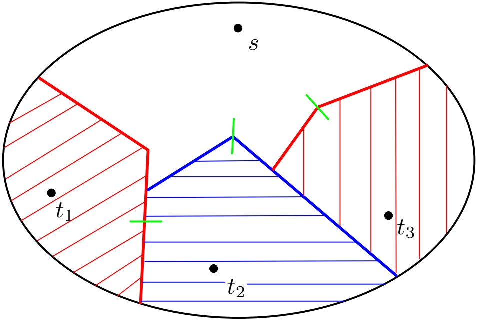
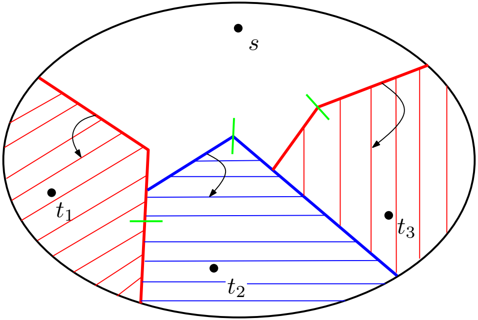


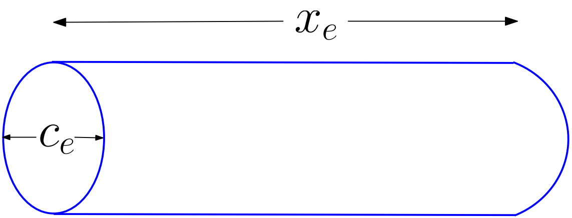
Construction Highlight: $k = 2$

- Total cost of the generated solution is no more than α times the value obtained by the linear program.
- We still need to consider whether proper connectivity is achieved in the residual graph.



Region Growing Lemma

- LP solution assigns length x_e to edges
- Surface area c_e
- Volume $x_e c_e$



Region Growing Lemma: Single Edge Length

- For well separated terminals s and t, there exists a cut C such that $cost(C) \leq \log h \times$ LP contribution contained inside C.

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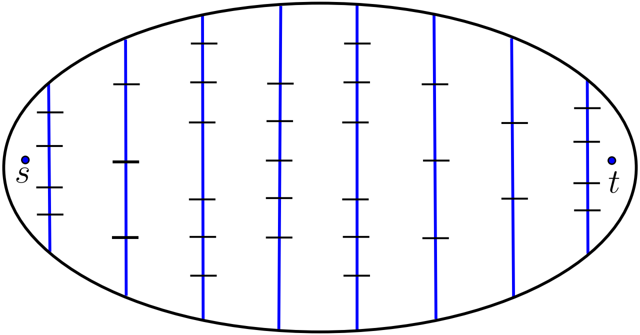

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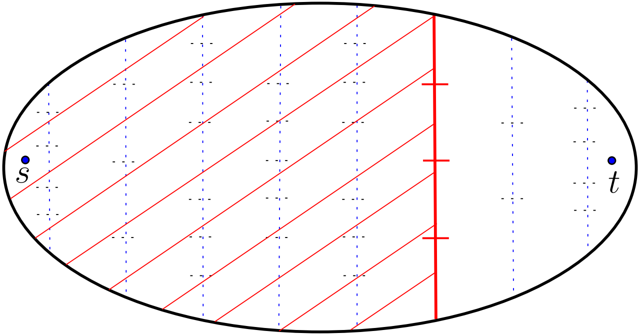
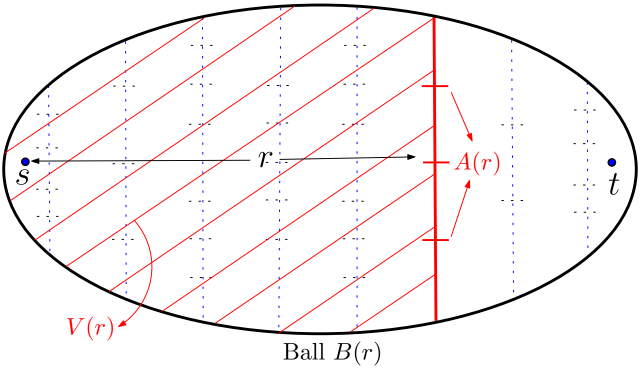


Region Growing Lemma: Single Edge Length

- $\exists r \in [0, 1]$ such that $A(r) \leq \log h \times V(r)$.



Region Growing Lemma for Multi-route Case

- Even with x and y lengths there exists a cut C such that $\#$ y edges in the cut small
- And x-$cost(C) \leq O(\log h) \times$ LP contribution (x's) contained inside C.

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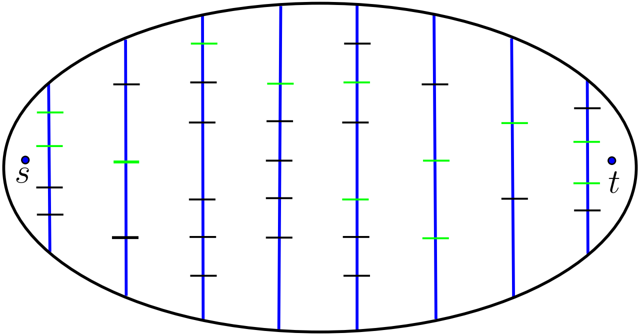

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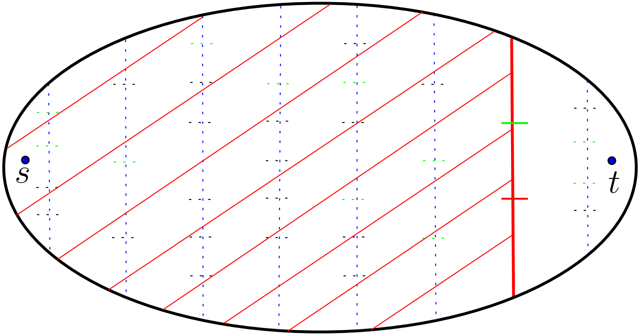
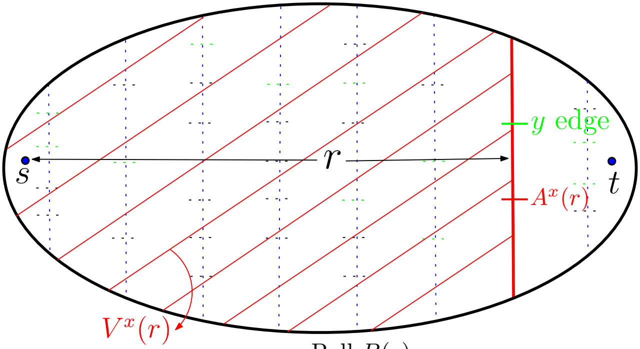


Region Growing Lemma for Multi-route Case

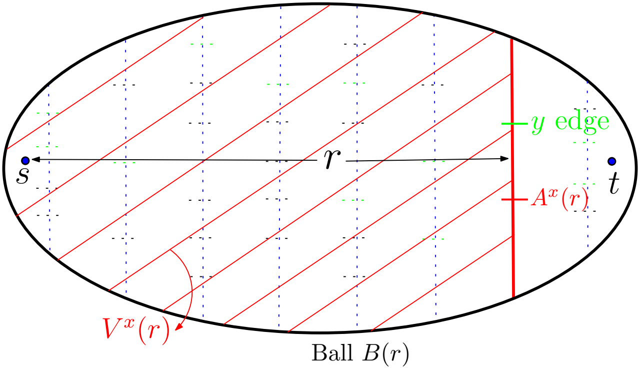
- $\exists r \in [0, 1]$ such that
  - $A^x(r) \leq 2 \log h \times V^x(r)$
  - $y(r) < 2(k-1)$
- Markov's Inequality



Ball $B(r)$

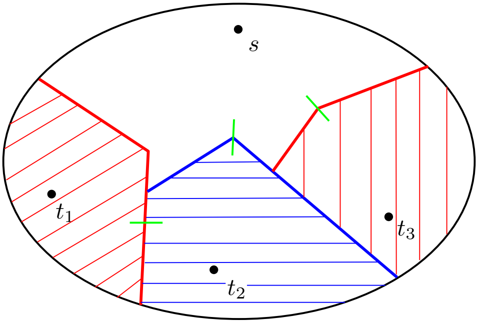
Region Growing Lemma for 2-route

- $\exists r \in [0,1]$ such that
  - $A^x(r) \leq 2\log h \times V^x(r)$
  - $y(r) < 2(k-1) = 2$



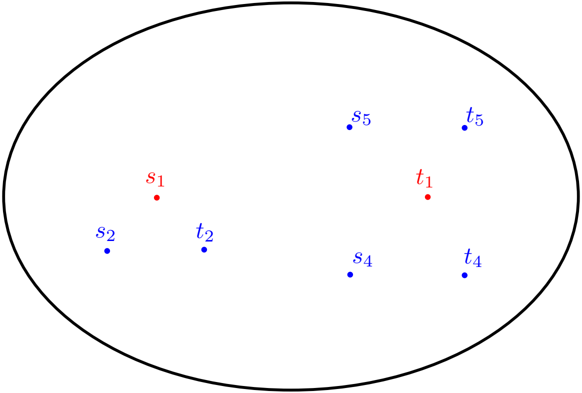
2-Route Cut Algorithms: Single Source Multiple Sink

- First we find a cut S that separates terminal, say t_1, from the source via region growing lemma.
- Then we then remove the cut from the graph and iterate over $G[V \setminus S]$.
- Overall this gives a $O(\log h)$ approximation.



2-Route Cut Algorithms: Multicut

- For 2-route multicuts in order to guarantee separation, after cut S is found by region growing, we need to recurse on $G[S]$ and $G[V \setminus S]$. This results in an $O(\log^2 h)$ approximation.



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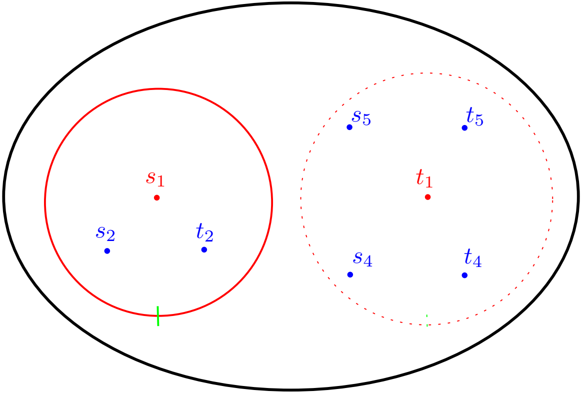

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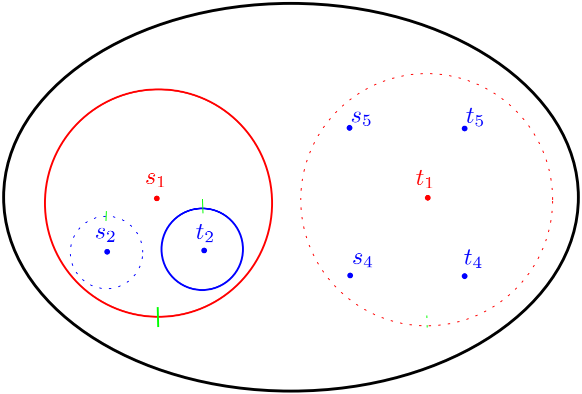
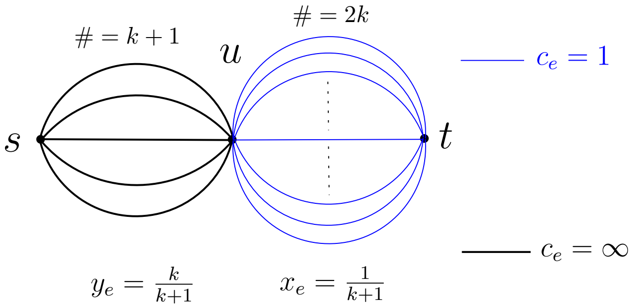


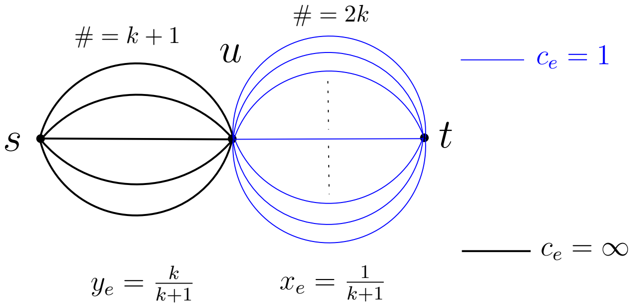
k-Route Cut Algorithms

- Integrality gap of the LP is $\Omega(k)$ for $k+1$ route cut



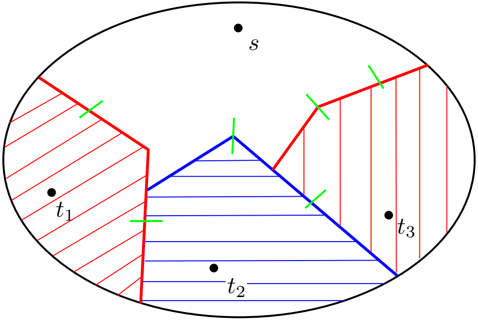
k-Route Cut Algorithms

- (α, β) approximation: s_i and t_i get $\alpha \times k$ separated with a cost of $\beta \times OPT$



k-Route Cut Algorithms

- Plainly applying the lemma we get $(2h, 2)$ and $(2, 2h)$
- Single Source Multiple Sink k-route cut using a separation idea over a strengthened LP we get a $(6, O(\sqrt{h}\log h))$ approximation.



Open Problems

- Poly-logarithmic (bicriteria) approximation for k-route cuts for $k \geq 3$.

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- Hardness of approximation for k-route cut problems.

Questions?