Secretary Problems with Convex Cost

Siddharth Barman

University of Wisconsin–Madison

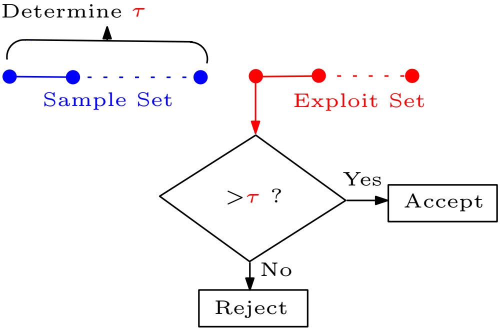
Joint work with Shuchi Chawla, David Malec, and Seeun Umboh

Classical Secretary Problem

- **Input:** Online stream of n elements in random order.
- The value of an element is revealed when it is received, and
- We need to make an irrevocable accept/reject decision.
- **Goal:** Select the highest-ranvalued element as often as possible, over all random orders.

Threshold rule

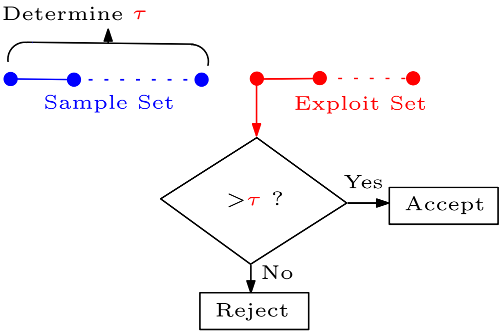
- Consider the first $n/2$ elements as sample set and unconditionally reject them all.
- Set threshold τ to be the maximum value in the sample set.
- Select the first element in the remaining stream with value higher than τ .



Threshold rule

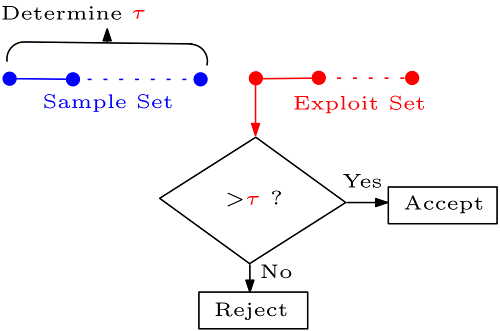
Good Events

- E_1 : Element with second highest rank in Sample Set.
- E_2 : Element with highest rank in Exploit Set.



Threshold rule achieves high success probability

- Since the elements are received in a random order probability that the highest ranked element is selected is at least $1/4$.
- Dynkin gave a threshold rule which selects the highest value with probability approaching $1/e$.



Motivating Examples

Many real-world resource allocation problems require online decision making.

- Wireless access point accepting requests from mobile nodes arriving one at a time.
  - It can accept at most a fixed number of nodes.
  - Goal: Accept nodes with high total value.
- Cloud Computing: Jobs with priority.
- Sponsored Search Auctions: Ad Slots.

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- Sponsored Search Auctions: Ad Slots.

In the examples above,

- more than one element can be selected and
- the objective was to maximize expected sum of values of selected elements.

Outline

- Background.
- Secretary Problem with Convex Costs (SPCC).
- Unit-Size SPCC.
- General SPCC.
- Constrained SPCC.

Competitive Analysis

Definition (Competitive Ratio)

An algorithm ALG is c-**competitive** if

- over random orderings σ of elements,
- it achieves an expected total value at least $1/c$ times the *optimal offline solution*.

$$\mathrm{E}_{\sigma}[\mathrm{ALG}(\sigma)] \geq \mathrm{OPT}/c.$$

For example, we say that the threshold rule for the classical secretary problem is e-competitive.

Feasibility Constraints

Generalizations which select multiple elements and maximize sum of values

- **Multiple-Choice Secretary:**
  - Can select up to k elements.
  - Constant competitive ratio: $1 + O(\sqrt{1/k})$ [Kleinberg 2005] .

Feasibility Constraints

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- **Knapsack Secretary:**
  - Each element has an associated size along with its value.
  - The size of the online selected subset is no more than a specified budget.
  - Constant competitive ratio [Babaioff et al. 2007].

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  - The size of the online selected subset is no more than a specified budget.
  - Constant competitive ratio [Babaioff et al. 2007].
- **Matroid Secretary:**
  - Selected set of elements is restricted to be an independent set.
  - Competitive ratio: $O(\sqrt{\log r})$ where r (rank) is the cardinality of the largest feasible set. [Chakraborty et al. 2012].

Wireless Access Point Example Revisited

Tradeoff:

- Wish to accept as many nodes as possible.
- But more power required maintain signal-to-noise ratio.
- Power is a non-linear (convex) function of the total load.

This directly translates into a value minus cost type of objective rather than a simple sum of values objective.

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Secretary Problem with Convex Cost (SPCC)

Definition (SPCC)

- A stream of n elements in random order.
- A convex cost function $f(\cdot)$.
- Each element in the stream has a value v_i and size s_i
- Goal: Select a set A online with maximum profit $\pi(A)$.

$$\pi(A) := v(A) - f(s(A))$$

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Special cases:

- Secretary with knapsack constraint.

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Primary Challenges

- Subsets with negative profit $\implies$ no trivial reduction to standard secretary.
- Initial bad decisions may drive cost too high to accept even high-valued elements later on.

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Theorem

There exists a constant-competitive algorithm for SPCC.

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Secretary Problem with Convex Cost: Unit-Size

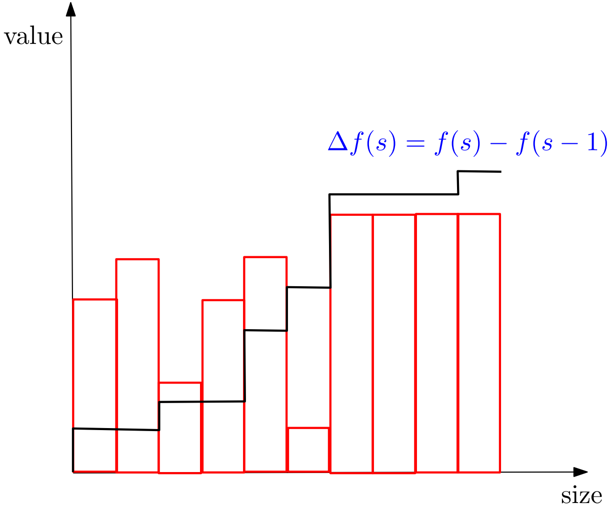
Setting:

- All elements have size 1.
- The cost function $f()$ depends only on the cardinality of the selected set.

The objective now is to select a set online which maximizes

$$\pi(A) = v(A) - f(|A|)$$

Unit-Size: Offline Algorithm



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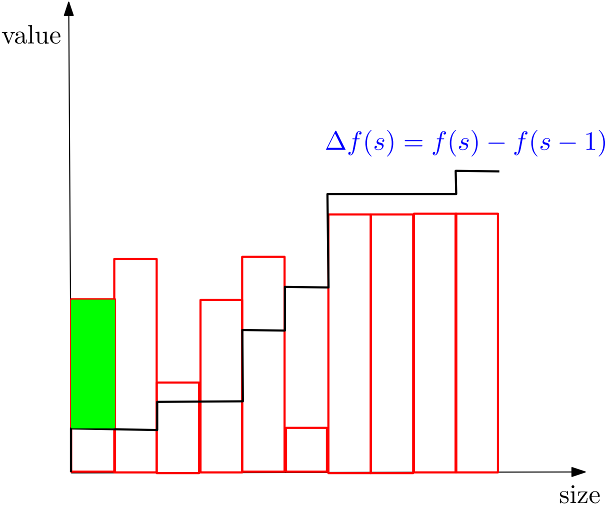

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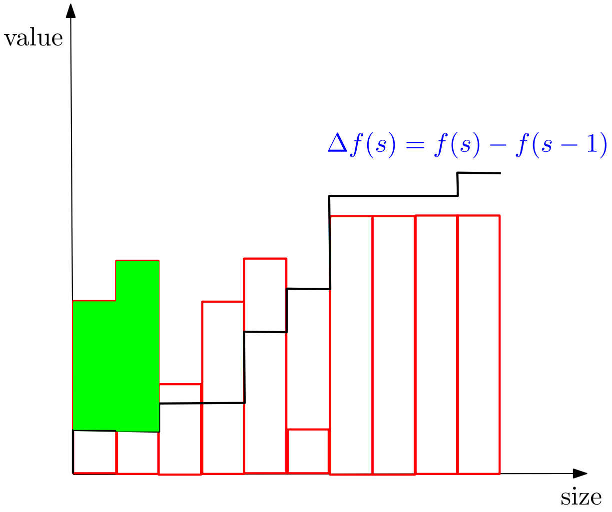

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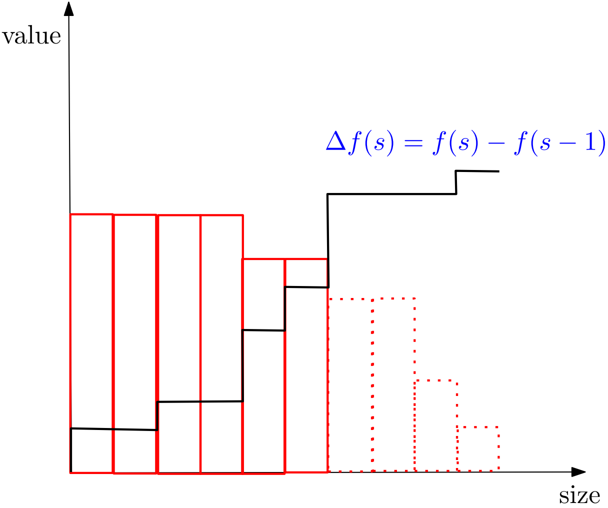

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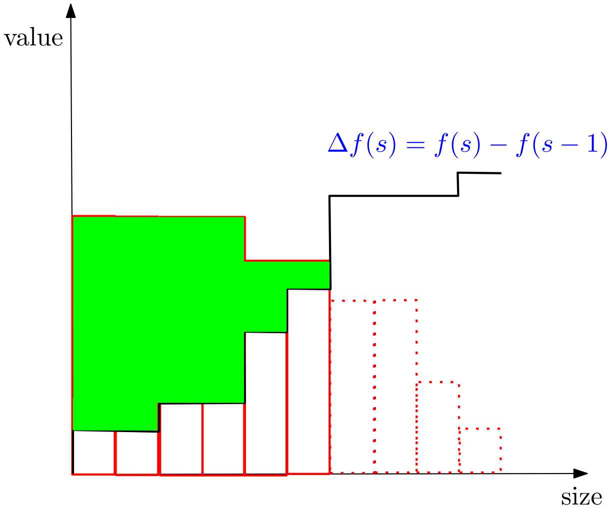

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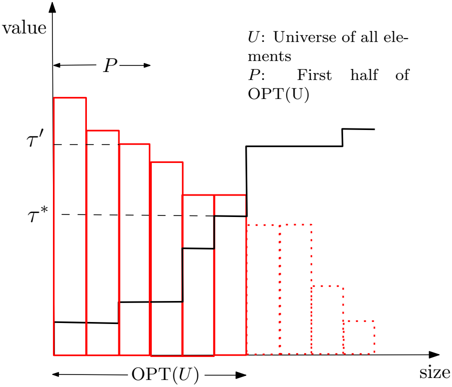
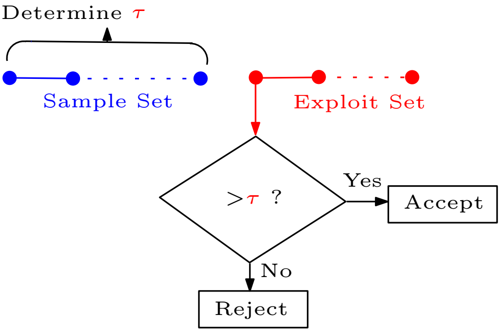


- Greedy solution gives us an ordering and an optimal cut-off value τ^*.
- It is sufficient to have threshold $\tau' \geq \tau \geq \tau^*$.

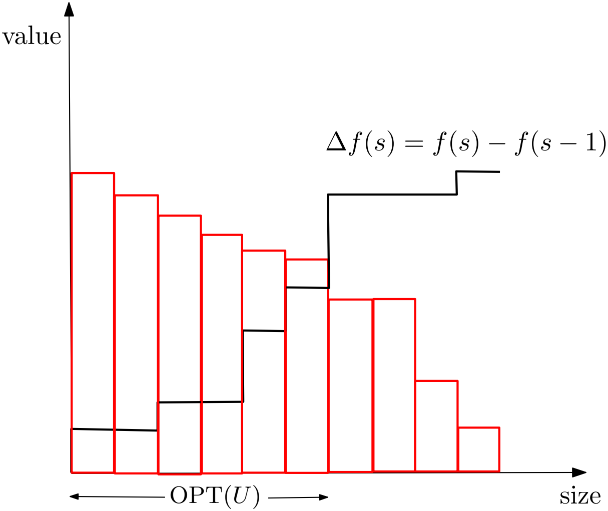
General Solution Strategy

High-level approach:

- Develop a greedy exact/approximate offline algorithm.
- Use greedy algorithm to determine threshold from sample set.
- Apply the threshold over remaining stream.



Threshold Selection



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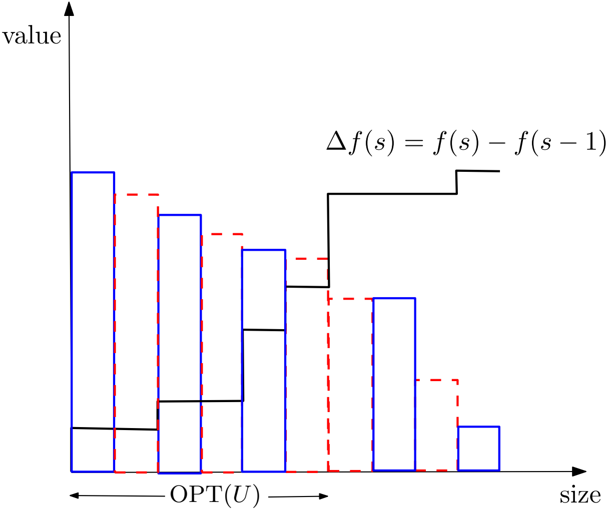

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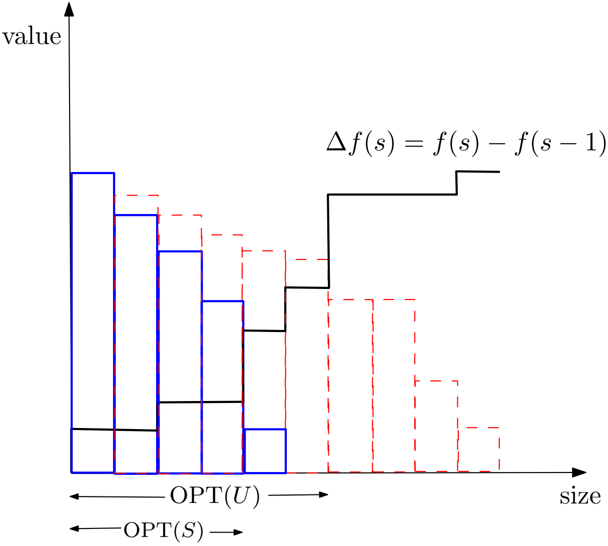

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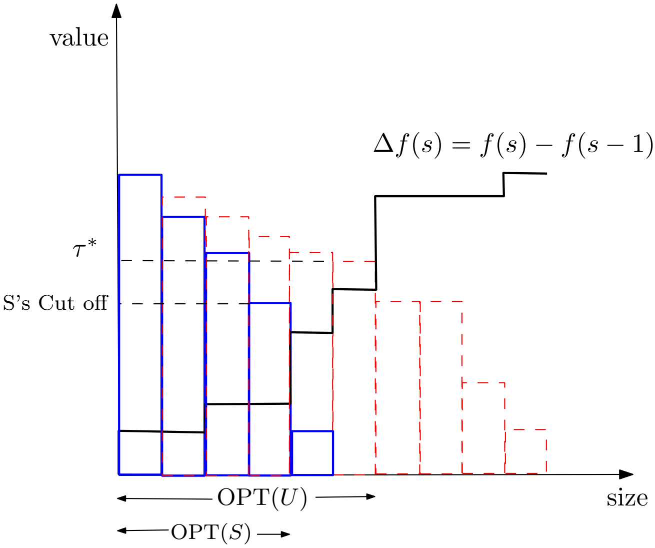

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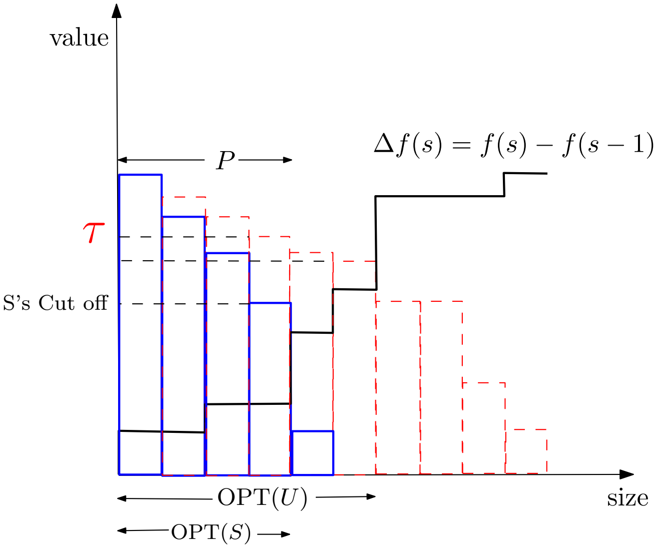

Unit-Size: Online

We use the greedy offline algorithm over a sample set to determine value threshold τ with a large enough value prefix set P.

Algorithm $\mathcal{A}$

1. Draw k from $\mathrm{Binomial}(n, 1/2)$ and let sample set S be the first k elements seen $\implies$ S is a random subset.
2. Use the greedy offline algorithm to compute $\mathrm{OPT}(S)$. Let $\ell = |\mathrm{OPT}(S)|$.
3. Set value threshold τ to the the $\ell/3$rd largest value in $\mathrm{OPT}(S)$.
4. For elements $k+1$ through n, select every element that has value at least τ and does not decrease profit.

Probabilistic Analysis

Good Events(Classical Secretary Problem)

- E_1 : Element with second-highest rank in Sample Set.
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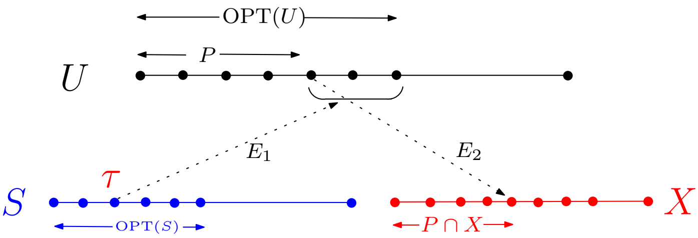
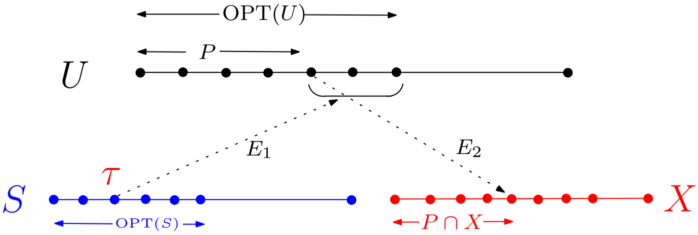
Good Events(SPCC)

- E_1 : Good threshold: $\tau' \geq \tau \geq \tau^*$.
- E_2 : Sufficient elements satisfying threshold in Exploit Set.



SPCC:Unit Size



We can use Chernoff bound to guarantee that $\Pr[E_1 \wedge E_2]$ is a fixed constant. Overall we get the desired result:

Theorem

Algorithm $\mathcal{A}$ achieves a constant competitive ratio for the setting with unit sizes.

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Secretary Problem with Convex Cost: Arbitrary Sizes

Recall High-Level Approach:

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- Use greedy algorithm to determine threshold from sample set.
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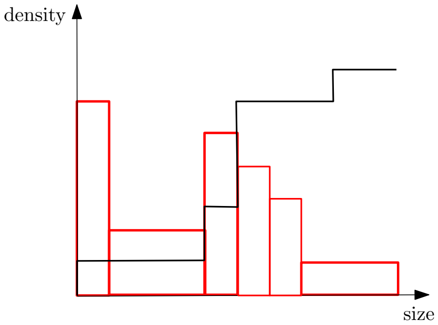
Arbitrary Sizes: Offline

- In this case value-ordering is suboptimal.
- Instead consider density := value/size.
- As before profit maximization translates into maximizing area enclosed between the curves.



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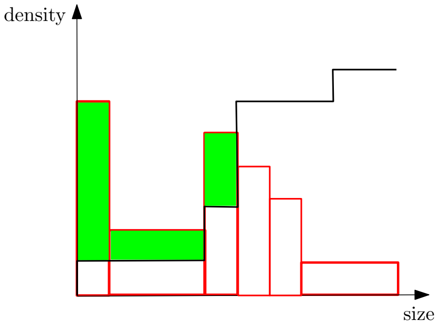

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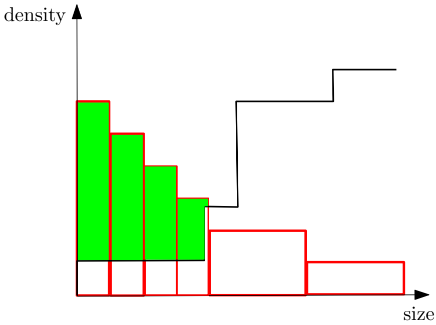


Arbitrary Sizes: Offline

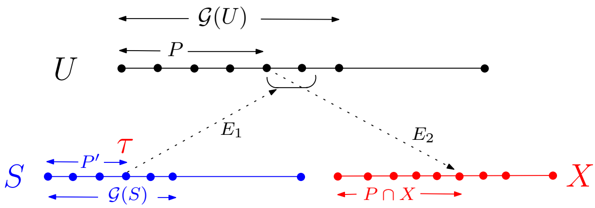
Density-based greedy selection

1. Sort elements in decreasing ordering of density.
2. $\mathcal{G}$: Select in order till marginal profit $\pi(\mathcal{G} + i) - \pi(\mathcal{G})$ is negative.
3. Output $\mathcal{G}$ or the single most profitable element, whichever has more profit.

Theorem

Density-based greedy selection achieves a 3-approximation for the offline setting.

Arbitrary Sizes: Online



- Target now is density prefix $\mathcal{G}(U)$.

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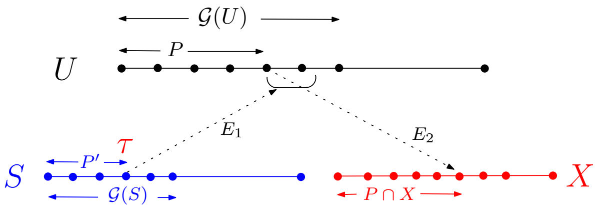

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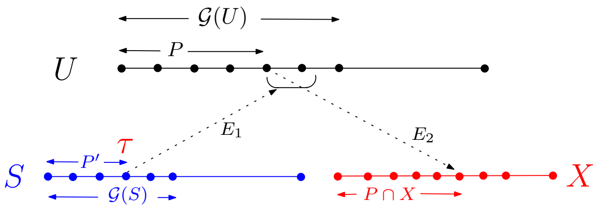
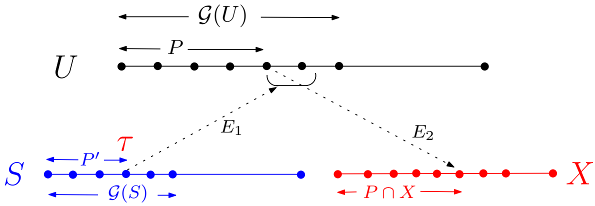


- Target now is density prefix $\mathcal{G}(U)$.
- Prefix set P has profit a constant times the profit of $\mathcal{G}(U)$.
- Density threshold τ selected such the density prefix set P' has profit a constant times $\mathcal{G}(S)$.

Arbitrary Sizes: Online



- E_1 : $\pi(S \cap \mathcal{G}(U)) \geq \alpha\pi(\mathcal{G}(U))$ implies good threshold.
- E_2 : $\pi(P \cap V) \geq \beta\pi(P)$ implies a good competitive ratio when τ is applied to V.

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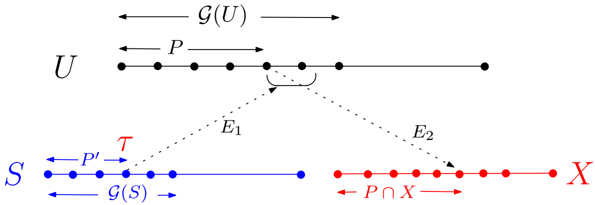

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- Chernoff bound implies $\Pr[E_1 \wedge E_2]$ is constant.

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Threshold algorithm achieves a constant competitive ratio for SPCC.

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Generalizations of Secretary with Convex Cost

Recall the generalizations of Classic Secretary Problem:

- Multiple-Choice Secretary Problem,
- Knapsack Secretary Problem and
- Matroid Secretary.

Multiple-Choice SPCC

Offline Problem:

- Density-based greedy algorithm may return a set of small total size $\implies$ low value.
- In particular, k densest elements may have small size each.

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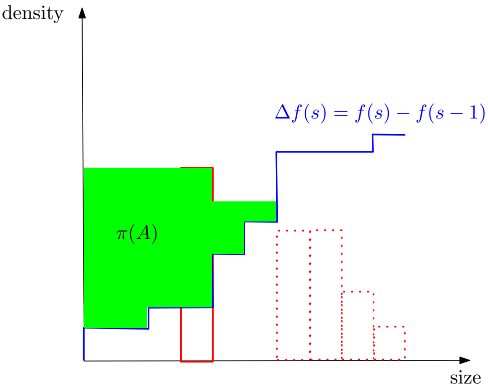
Main Idea

1. Reduce from value minus cost objective to sum of value objective.
2. Black-box reduction from multiple-choice SPCC to multiple-choice standard secretary.

Profit Decomposition



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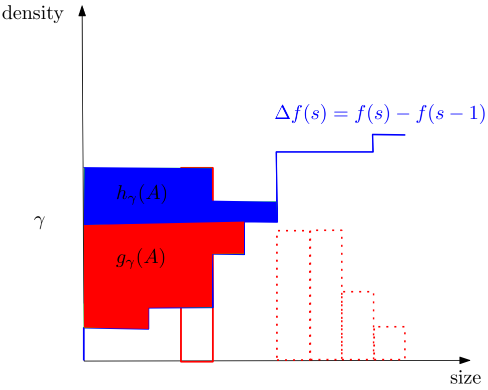

$$\pi(A) = h_\gamma(A) + g_\gamma(A)$$

Reduction to Standard Multiple-Choice Secretary

For any sufficiently high γ,

1. if we ignore elements of density less than γ,
2. maximizing $g_\gamma(A)$ and $h_\gamma(A)$ is as easy as sum of values maximization $\Longrightarrow$ choose k best, and
3. Let $G = \mathrm{argmax}_A\, g_\gamma(A)$ and $H = \mathrm{argmax}_A\, h_\gamma(A)$.

$$\begin{aligned} \pi(\mathrm{OPT}) &= g_\gamma(\mathrm{OPT}) + h_\gamma(\mathrm{OPT}) \\ &\leq g_\gamma(G) + h_\gamma(H) \\ &\leq \pi(G) + \pi(H). \end{aligned}$$

Reduction to Standard Multiple-Choice Secretary

Online

1. Compute a sufficiently high density threshold τ.
2. Choose at random: h_τ or g_τ.
3. Run standard multiple-choice secretary algorithm on Exploit Set, ignoring elements below threshold.

We can define similar events E_1 and E_2 as before.

Reduction to Standard Matroid Maximization

Online

1. Compute a sufficiently high density threshold τ.
2. Choose at random: h_τ or g_τ.
3. Run standard matroid secretary algorithm on Exploit Set, ignoring elements below threshold.

We can define similar events E_1 and E_2 as before.

Summary of Results

Setting	Competitive Ratio
Unconstrained	Constant
Matroid	Constant times α
ℓ-dimensional costs	Constant times ℓ
Matroid + ℓ-dimensional costs	Constant times $\alpha \cdot \ell$

where α is the competitive ratio of the standard matroid secretary algorithm that we used as a black-box.

Questions?

Thanks!