CS412 Spring Semester 2011

Homework Assignment #1

Due Thursday February 3rd 2011, in class

Sum of all problems : 120%, Maximum possible score : 100%.

For the following problems, even if you do not answer a given question, you are still allowed to use its result in order to answer a subsequent question.

1. [30%] Examine if the function $g(x)$ is a contraction in the following cases:
   (a) $g(x) = 2x + 3, \quad x \in \mathbf{R}$
   (b) $g(x) = 2\sin(x/3), \quad x \in \mathbf{R}$
   (c) $g(x) = 0.5x + c, \quad x \in \mathbf{R}$ where c is an arbitrary number

2. [20%] Each of the following examples describes a fixed point iteration and a nonlinear equation. In each case, *assuming that the fixed point iteration will converge*, show that the limit is a solution of the respective equation.
   (a) The iteration $x_{k+1} = \dfrac{ax_k^2 - c}{2ax_k + b}$ and the equation $ax^2 + bx + c = 0$.
   (b) The iteration $x_{k+1} = \dfrac{3x_k^2 + a}{4x_k}$ and the equation $x^2 - a = 0$.

3. [20%] For each of the 2 nonlinear equations described in question 2 above, examine whether the given fixed point iteration is the same as Newton's method for that equation.

4. [50%] Our initial analysis of Newton's method assumed that $f'(a) \neq 0$, where a is the solution of the nonlinear equation. This exercise will investigate what happens in the special case where $f'(a) = 0$ but $f''(a) \neq 0$.
   (a) [20%] Let a be a solution of $f(x) = 0$, and assume $f'(a) = 0$ and the second derivative f'' is continuous and $f''(a) \neq 0$. Show that:
   $$\lim_{x\to a} \frac{f(x)}{[f'(x)]^2} = \frac{1}{2f''(a)}$$
   [Hint: You may want to use L'Hôspital's rule which states that for the differentiable functions $\alpha(x)$, $\beta(x)$:
   $$\textit{If } \lim_{x\to x_0} \alpha(x) = \lim_{x\to x_0} \beta(x) = 0 \textit{ and the limit } \lim_{x\to x_0} \frac{\alpha'(x)}{\beta'(x)} \textit{ exists,}$$
   $$\textit{then } \lim_{x\to x_0} \frac{\alpha(x)}{\beta(x)} = \lim_{x\to x_0} \frac{\alpha'(x)}{\beta'(x)} \quad]$$
   (b) [30%] Show that in this case, the iteration function $g(x) = x - \frac{f(x)}{f'(x)}$ generated by Newton's method is a contraction.