CS412 Spring Semester 2011

Homework Assignment #4

Due Tuesday April 5th 2011, in class

Sum of all problems : 120%, Maximum possible score : 100%.

1. [40%] In this problem you will use `MATLAB` to assess in practice the problems that arise when attempting to solve an ill-conditioned linear system of equations. For each of the values $k = 5, 6, 7, \ldots, 14, 15$ perform the following:

   - Form the Hilbert matrix $\mathbf{H}_n$ with entries $[H_n]_{ij} = \frac{1}{i+j-1}$. You can use the `MATLAB` function `hilb(k)` to generate this matrix.
   - Define the test vector
   $$\mathbf{x}_k = \begin{pmatrix} \sin(1) \\ \sin(2) \\ \vdots \\ \sin(k) \end{pmatrix}.$$
   - Construct the vector $\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k$
   - Compute the vector $\mathbf{z}_k = \mathbf{H}_k^{-1} \mathbf{y}_k$ using the `MATLAB` backslash operator (`\`). In theory, $\mathbf{z}_k$ should be identical to $\mathbf{x}_k$, but this will not be the case here, due to numerical errors.
   - Compute the absolute error $e_k = \|\mathbf{x}_k - \mathbf{z}_k\|_\infty$. You can use the `MATLAB` function `norm(v,inf)` to compute the infinity norm of a vector `v`.

   Plot the error values e_k as a function of the matrix size k. Use a logarithmic axis for the error, to make this plot easier to parse; the `MATLAB` function `semilogy` can be used for this purpose: The syntax is the same as the `plot` function, but it generates a plot with a logarithmic y-axis.

   Turn in a printout of this plot, along with the code used to generate it.

2. [40%] In this problem you will use `MATLAB` to compute the condition number of Vandermonde matrices of various dimensions. Remember that given n values $x_1, x_2, \ldots, x_n$, the associated Vandermonde matrix is defined as

$$\mathbf{V}(x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2^2 & x_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n^2 & x_n & 1 \end{bmatrix}$$

   To compute such matrices, you can use the `MATLAB` function `vander(x)`, which returns the Vandermonde matrix associated with the values in vector `x` (`x` should be properly initialized to contain the values $x_1, x_2, \ldots, x_n$). Perform the following tests:

- For all integer values of n between 2 and 12, define a set of n equally spaced x-values $x_1, x_2, \ldots, x_n$, such that $x_k = k$. Plot the condition number $\kappa_\infty(\mathbf{V})$ of the associated Vandermonde matrix as a function of n, using a logarithmic plot. The `MATLAB` function `cond(V,inf)` can be used to compute the condition number $\kappa_\infty(\mathbf{V})$.
- Repeat the previous experiment, but now set $x_k = k/n$ (i.e. the x values are still equally spaced, but placed closer togther). Generate the logarithmic plot of condition number as a function of n and compare with the previous case.
- Repeat the previous experiment, but now generate x values that are no longer equally spaced. Specifically, set each $x_k = k^2$. Plot the condition number as a function of n as before.
- In this last example, we shall use a fixed number of x values (5, in particular), defined as follows:

  $$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 4, \quad x_5 = 4 + \epsilon$$

  In this case you should plot the condition number as a function of the parameter ϵ. Use values of ϵ between 0.0001 and 1 and generate a doubly-logarithmic plot using the `MATLAB` function `loglog`.

  Note: You can create a set of $n+1$ logarithmically spaced values of ϵ between 0.0001 and 1 as `e=10.^(-4:4/n:0)`.

Describe your conclusions from these experiments. Can we identify certain scenarios when using the Vandermonde system for polynomial interpolation is problematic?

3. [40%] Prove the following properties (on paper):

   (a) Show that for any vector $\mathbf{x} \in \mathbf{R}^n$, the following inequalities hold:

   $$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty$$

   $$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$$

   (b) Assume that positive constants c_1, c_2 exist, such that for any $\mathbf{x} \in \mathbf{R}^n$

   $$c_1\|\mathbf{x}\|_a \leq \|\mathbf{x}\|_b \leq c_2\|\mathbf{x}\|_a$$

   Here, $\|\cdot\|_a$ and $\|\cdot\|_b$ are simply two different vector norms. Show that in this case, we can also find positive constants d_1, d_2 such that

   $$d_1\|\mathbf{M}\|_a \leq \|\mathbf{M}\|_b \leq d_2\|\mathbf{M}\|_a$$

   for any *matrix* $\mathbf{M} \in \mathbf{R}^{n\times n}$. The norms in the last expression are the matrix norms induced from the respective vector norms.