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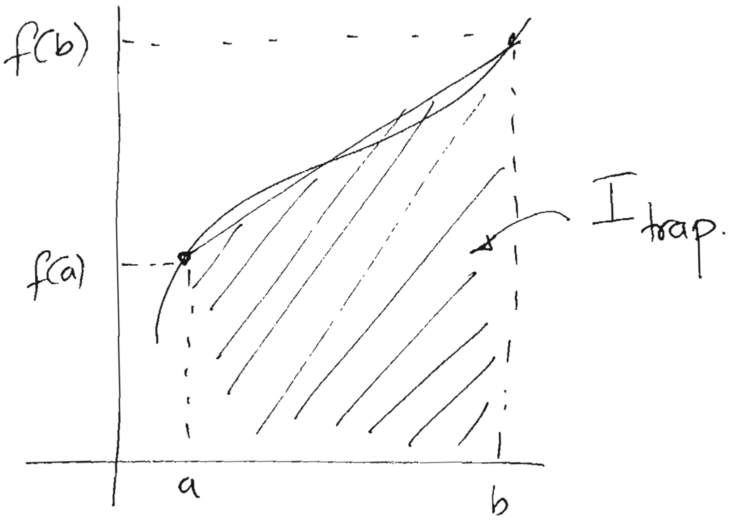
Trapezoidal rule

In this case f is approximated in $[a,b]$ with the straight line drawn between $(a, f(a))$ and $(b, f(b))$.



$$I_{trap} = \frac{f(a)+f(b)}{2} \cdot (b-a)$$

Thus in this case $I = \int_a^b f(x)dx \approx (b-a)\underbrace{\frac{f(a)+f(b)}{2}}_{=I_{trap}}$

To generate the corresponding composite rule, we write:

$$I_K = \int_{x_K}^{x_{K+1}} f(x)dx \approx (x_{K+1}-x_K)\frac{f(x_K)+f(x_{K+1})}{2} = h_K \frac{f(x_K)+f(x_{K+1})}{2} = I_{K,}$$

Thus $I = \sum_{K=0}^{N-1} I_K \approx \sum_{K=0}^{N-1} I_{K,trap} \overset{(h=const)}{=} \sum_{K=0}^{N-1} h \frac{f(x_K)+f(x_{K+1})}{2}$

$$= \frac{b-a}{2N}\left[f(x_0)+2f(x_1)+2f(x_2)+\dots+2f(x_{N-2})+2f(x_{N-1})+f(x_N)\right]$$

Note that due to the simple formula for 4/4/11 L

the trapezoidal area, we did not have to write the approximating polynomial $p(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ explicitly. Also, the result of integrating $\int_a^b p(x)\,dx$ results in a very simple formula $\left[(b-a)\frac{f(a)+f(b)}{2} \right]$, even "simpler" than the formula for p itself!

<u>Local error analysis</u>:

Estimating the local error can be somewhat delicate with the trapezoidal rule ... we will in this case use a formula from the theory of interpolating polynomials we saw before:

<u>Thm</u> If $p(x)$ is a n-degree polynomial, interpolating $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, then for every $x \in [x_0, x_n]$ we have

$$f(x) - p(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

<u>CAUTION</u> c is <u>not</u> a constant; it depends on the particular $\underline{\underline{x}}$ we chose in this theorem.

For the trapezoidal rule, we effectively use a linear $(n=1)$ interpolant, thus when $x \in [x_K, x_{K+1}]$:

$$f(x) - p^K(x) = \frac{f''(c_K)}{2}(x-x_K)(x-x_{K+1})$$

where $c_K \in (x_K, x_{K+1})$

$$e_K = \left| \int_{x_K}^{x_{K+1}} [f(x) - p^{(K)}(x)]\, dx \right| =$$

$$= \left| \int_{x_K}^{x_{K+1}} \frac{f''(c_K)}{2}(x-x_K)(x-x_{K+1})\, dx \right|$$

$$\leq \int_{x_K}^{x_{K+1}} \left| \frac{f''(c_K)}{2} \right| \left| (x-x_K)(x-x_{K+1}) \right| dx$$

$$\leq \frac{1}{2} \| f''(x) \|_\infty \int_{x_K}^{x_{K+1}} \left| (x-x_K)(x-x_{K+1}) \right| dx$$

The only reason we can meaningfully continue at this point, is to recognize that $(x-x_K)(x-x_{K+1}) \leq 0$ in $[x_K, x_{K+1}]$ thus $|(x-x_K)(x-x_{K+1})| = -(x-x_K)(x-x_{K+1})$ and remove in this way the absolute value in the integral above.

This is not the case in general for higher-order polynomial interpolants, where we won't be able to remove the absolute value. (see Simpson's rule next).

We can verify that: 4/14/11

$$\int_{x_K}^{x_{K+1}} |(x-x_K)(x-x_{K+1})| = -\int_{x_K}^{x_{K+1}} (x-x_K)(x-x_{K+1}) = \frac{h_K^3}{6}$$

Putting everything together:

$$e_K \leq \frac{1}{2} \|f''\|_\infty \cdot \frac{h_K^3}{6} \Rightarrow \boxed{e_K \leq \frac{1}{12} \|f''\|_\infty \cdot h_K^3}$$

For the global error:

$$e \leq \sum_{K=0}^{N-1} e_K \leq \frac{N}{12} \|f''\|_\infty h_K^3 \xrightarrow{Nh=b-a} \boxed{e \leq \frac{b-a}{12} \|f''\| h^2}$$

Thus, trapezoidal rule is also 2nd order accurate.

SIMPSON's rule This is a slightly more complicated algorithm, but the accuracy gains are so attractive that it has become somewhat of a golden standard for numerical integration.

It is based on (piecewise) quadratic interpolation.

Specifically:

Consider 3 equally spaced x-values

x_1 , $x_2 = x_1 + h$, $x_3 = x_1 + 2h$

with associated y-values $y_i = f(x_i)$ $i = 1,2,3$.

We will approximate $f(x)$ in $[x_1, x_3]$ with a <u>quadratic</u> $p(x) = c_2 x^2 + c_1 x + c_0$ that interpolates the 3 data points: $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

Using Lagrange interpolation:

$$l_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-x_2)(x-x_3)}{(-h)(-2h)} = \frac{1}{2h^2}(x-x_2)(x-x_3)$$

$$l_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = -\frac{1}{h^2}(x-x_1)(x-x_3)$$

$$l_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{1}{2h^2}(x-x_1)(x-x_2)$$

And $p(x) = y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x)$

We then proceed to approximate

$$\underbrace{\int_{x_1}^{x_3} f(x)\,dx}_{I} \approx \underbrace{\int_{x_1}^{x_3} p(x)\,dx}_{I_{simp.}} = \sum_{i=1}^{3} y_i \int_{x_1}^{x_3} l_i(x)\,dx$$

After some easy (yet tedious) analytic integration using the previous formulas, we get:

$$\int_{x_1}^{x_3} l_1(x)\,dx = \frac{h}{3}, \quad \int_{x_1}^{x_3} l_2(x)\,dx = \frac{4h}{3}, \quad \int_{x_1}^{x_3} l_3(x)\,dx = \frac{h}{3}$$

Thus $\displaystyle\int_{x_1}^{x_3} p(x)\,dx = \frac{h}{3}\left[f(x_1) + 4f(x_2) + f(x_3)\right]$.

This is Simpson's rule, and is commonly written as:

$$\int_{a}^{b} f(x)\,dx \approx \frac{\overbrace{b-a}^{=2h}}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

In order to define the respective composite rule, we use a partitioning: 4/14/11

$$a = x_0 < x_1 < x_2 < \cdots < x_{2N-1} < x_{2N} = b$$

this time we define each interval $D_k = [x_{2k}, x_{2(k+1)}]$, and

$$I_k = \int_{x_{2k}}^{x_{2k+2}} f(x)\,dx \quad \Rightarrow \quad I = \sum_{k=0}^{N-1} I_k.$$

Then: $$I_{k,simp} = \frac{h}{3}\left[f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\right]$$

and the composite rule $I_{simp} = \sum_{k=0}^{N-1} I_{k,simp}$ becomes:

$$I_{simp} = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots$$

$$\cdots + 2f(x_{2N-4}) + 4f(x_{2N-3}) + 2f(x_{2N-2}) + 4f(x_{2N-1}) + f(x_{2N})\Big]$$

In order to estimate the local error, we again try to use the formula for the interpolation error of passing a quadratic $p^{(k)}(x)$ through $(x_{2k}, f(x_{2k}))$, $(x_{2k+1}, f(x_{2k+1}))$ and $(x_{2k+2}, f(x_{2k+2}))$.

$$f(x) - p^{(k)}(x) = \frac{f'''(c_k)}{3!}(x-x_{2k})(x-x_{2k+1})(x-x_{2k+2})$$

thus

$$e_k = \left| \int_{x_{2k}}^{x_{2k+2}} [f(x) - p(x)]\,dx \right|$$

$$\leq \int_{x_{2k}}^{x_{2k+2}} \frac{|f'''(c_k)|}{3!} \left|(x-x_{2k})(x-x_{2k+1})(x-x_{2k+2})\right| dx$$

$$\leq \frac{1}{6} \|f'''(x)\|_\infty \int_{x_{2k}}^{x_{2k+2}} \underbrace{\left|(x-x_{2k})(x-x_{2k+1})(x-x_{2k+2})\right|}_{(*)} dx$$

And with this we're at a dead end! We cannot simply remove the absolute value in the expression $(*)$ since it changes sign in $[x_{2k}, x_{2k+2}]$. Even if we break up this integral in sub-intervals, we will <u>at best</u> show that Simpson's rule is 3rd order accurate, where in fact it is even <u>more</u>, i.e. 4th order accurate!

To achieve our goal, we will use a different (and more general) type of analysis:

It is possible to show that:

<u>Thm</u> If an integration rule integrates <u>exactly</u> any polynomial up to degree $(d-1)$ then the global error is $O(h^d)$ or better, i.e. the rule is at least d-order accurate.

<u>Methodology</u> We will Test Simpson's rule on monomials $f(x) = x^d$, $d = 0, 1, 2, \ldots$

- $f(x) = 1$: $I_{simp} = \frac{b-a}{6}[1+4+1] = (b-a) \equiv \int_a^b 1 \cdot dx$

CORRECT!

- $f(x) = x$: $I_{simp} = \frac{b-a}{6}\left[a + 4\left(\frac{a+b}{2}\right) + b\right] =$

$$= \frac{b^2}{2} - \frac{a^2}{2} \equiv \int_a^b x\,dx \quad \text{CORRECT!}$$

- $f(x) = x^2$: $I_{simp} = \frac{b-a}{6}\left[a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right] =$

$$= \frac{b^3}{3} - \frac{a^3}{3} \equiv \int_a^b x^2\,dx \quad \text{CORRECT!}$$

$\cdot f(x) = x^3$: $I_{simp} = \frac{b-a}{6}\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right] =$

$$= \frac{b^4}{4} - \frac{a^4}{4} \equiv \int_a^b x^3\,dx \quad \text{CORRECT!}$$

$f(x) = x^4$: $I_{simp} = \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right]$

which is not equal to $\frac{b^5}{5} - \frac{a^5}{5} = \int_a^b x^4\,dx$

Thus Simpson's rule is 4th order accurate

i.e. $e_{k,local} \leq O(h^5)$

$e_{global} \leq O(h^4)$