The Newton interpolation method for the data points

$$(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$$

describes the n-degree polynomial interpolant as

$$
\begin{aligned}
p(x) = \; & c_0 \cdot \underbrace{1}_{n_0(x)} \\
& + c_1 \cdot \underbrace{(x-x_0)}_{n_1(x)} \\
& + c_2 \underbrace{(x-x_0)(x-x_1)}_{n_2(x)} \\
& \vdots \\
& + c_n \underbrace{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}_{n_n(x)}
\end{aligned}
$$

The coefficients $\{c_i\}$ are computed using the method of divided differences: For a set of x-values $x_i, x_{i+1}, \dots, x_j$ we define the divided difference as $f[x_i, \dots, x_j]$.

The value of this symbol is recursively defined as:

$\rightarrow$ $f[x_i] = y_i$ for $i = 0, \dots, n$

$\rightarrow$ $$f[x_i, x_{i+1}, \dots, x_j] = \frac{f[x_{i+1}, \dots, x_j] - f[x_i, x_{i+1}, \dots x_{j-1}]}{x_j - x_i}$$

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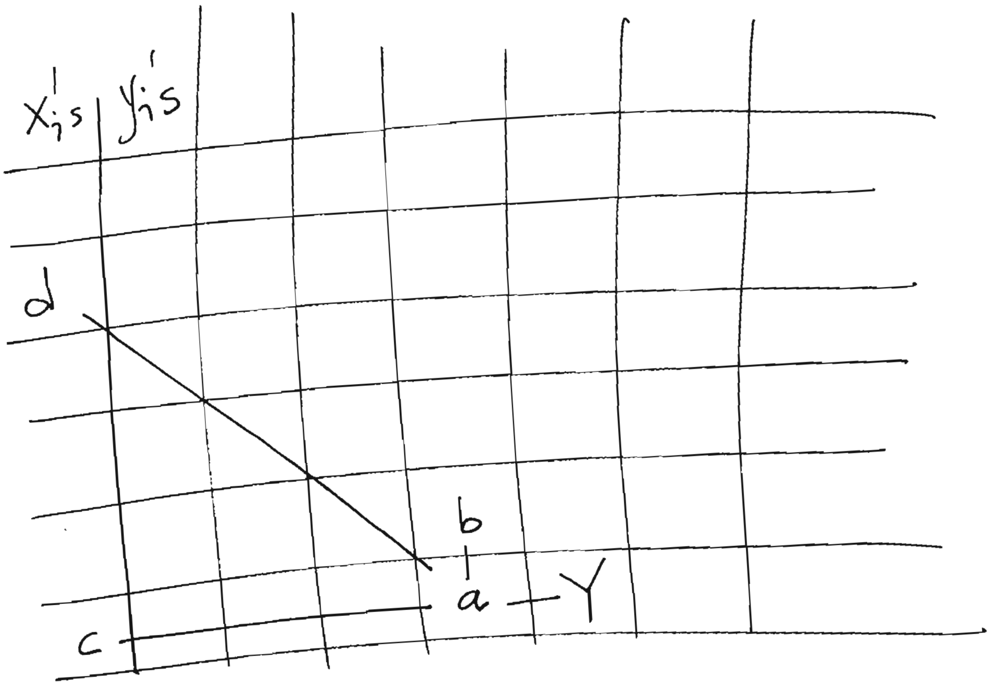
Once all relevant divided differences have been computed, the coefficients c_i are given by:

$$c_0 = f[x_0]$$
$$c_1 = f[x_0, x_1]$$
$$c_2 = f[x_0, x_1, x_2]$$
$$\vdots$$
$$c_n = f[x_0, x_1, \ldots, x_n]$$

Divided differences are usually tabulated as follows

	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	
x_0	$\boxed{f[x_0]} \to c_0$			
x_1	$f[x_1]$	$\boxed{f[x_0,x_1]} \to c_1$		
x_2	$f[x_2]$	$f[x_1,x_2]$	$\boxed{f[x_0,x_1,x_2]} \to c_2$	
x_3	$f[x_3]$	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$\cdots$
x_4	$f[x_4]$	$f[x_3,x_4]$	$f[x_2,x_3,x_4]$	

The recursive definition can be implemented directly on the table as follows



$$Y = \frac{a-b}{c-d}$$

e.g $(x_0, y_0) = (-2, -27)$ $(x_1, y_1) = (0, -1)$ $(x_2, y_2) = (1, 0)$

c_0 c_1 c_2

x_i's	y_i's			
-2	-27			
0	-1	13		
1	0	1	-4	

Easy evaluation

e.g. $$\begin{aligned} p(x) = {} & c_0 \\ & + c_1 (x-x_0) \\ & + c_2 (x-x_0)(x-x_1) \\ & + c_3 (x-x_0)(x-x_1)(x-x_2) \\ & + c_4 (x-x_0)(x-x_1)(x-x_2)(x-x_3) = \end{aligned}$$

$$= \underbrace{c_0 + (x-x_0)\underbrace{\Big[c_1 + (x-x_1)\underbrace{\Big[c_2 + (x-x_2)\underbrace{\Big[c_3 + (x-x_3]\,\underbrace{c_4}_{Q_4(x)}\Big]}_{Q_3(x)}\Big]}_{Q_2(x)}\Big]}_{Q_1(x)}}_{Q_0(x)}$$

$$P(x) = Q_0(x)$$

recursively: $$Q_n(x) = c_n$$

$$Q_{n-1}(x) = c_{n-1} + (x - x_{n-1})\, Q_n(x)$$

The value of $P(x) = Q_0(x)$ can be evaluated (in linear time) by iterating this recurrence n times

We also have:

$$Q_{n-1}(x) = c_{n-1} + (x - x_{n-1})\, Q_n(x)$$

$$\Rightarrow Q_{n-1}'(x) = Q_n(x) + (x - x_{n-1})\, Q_n'(x)$$

Thus, once we have computed all the Q_k's we can also compute all the derivatives too. Ultimately, $P'(x) = Q_0'(x)$.