We saw 3 methods for polynomial interpolation (Vandermonde, Lagrange, Newton). It is important to understand that all 3 methods compute (in theory) the same exact interpolant $p(x)$, just following diffent paths which may be better or worse from a computational perspective.

The question however, remains:

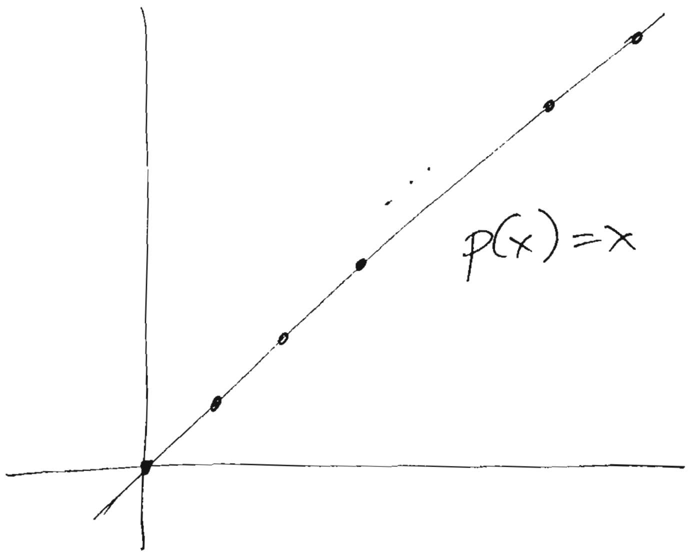
→ How accurate is this interpolation

or, in other words

→ How close is $p(x)$ to the "real" function $f(x)$!

Example:

$(x_0, y_0) = (0,0)$
$(x_1, y_1) = (1,1)$
$\vdots$
$(x_n, y_n) = (n, n)$



Using Lagrange polynomials $P(x)$ $(=x)$ is written as

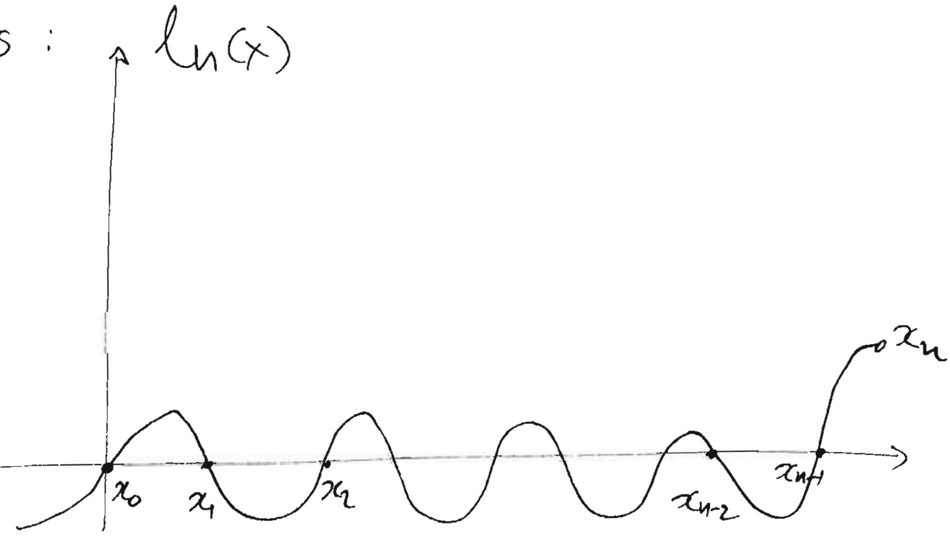
$$P(x) = \sum_{i=0}^{n} y_i \, l_i(x)$$

Let us "shift" y_n by a small amount δ. The new value is $y_n^* = y_n + \delta$. The updated interpolant $P^*(x)$ then becomes:

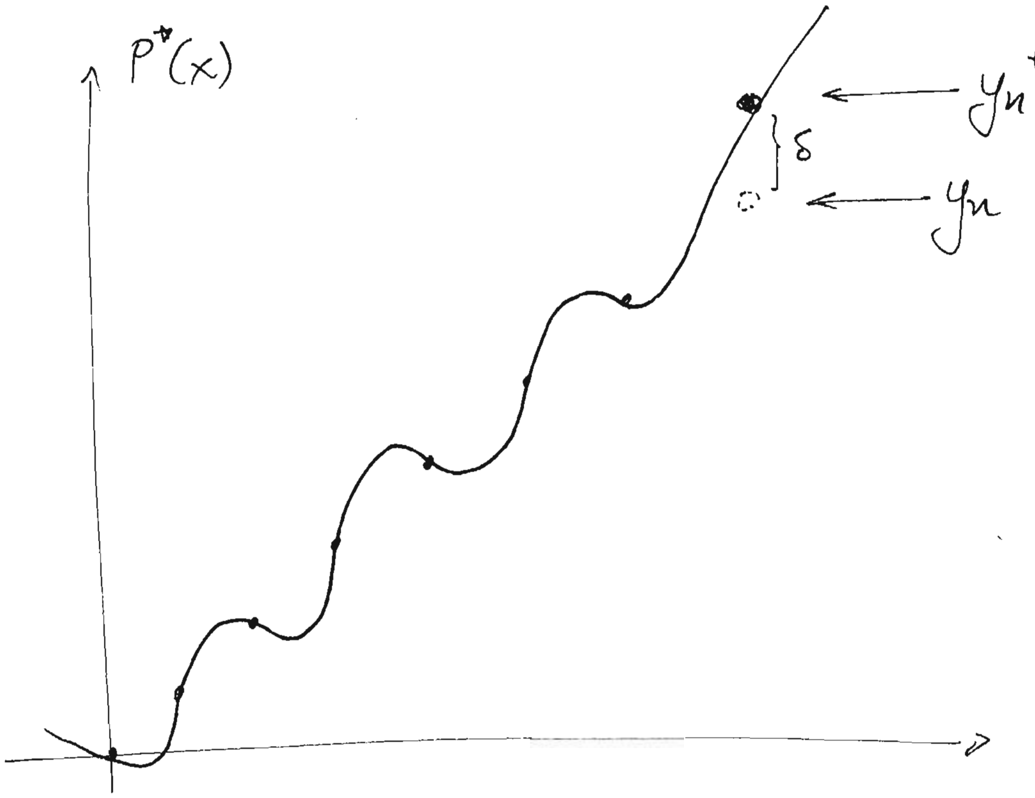
$$P^*(x) = \sum_{i=0}^{n-1} y_i \, l_i(x) + y_n^* \, l_n(x)$$

Thus $P^*(x) - P(x) = \delta \cdot l_n(x)$.

Note that l_n is a function that "oscillates" through zero several times:

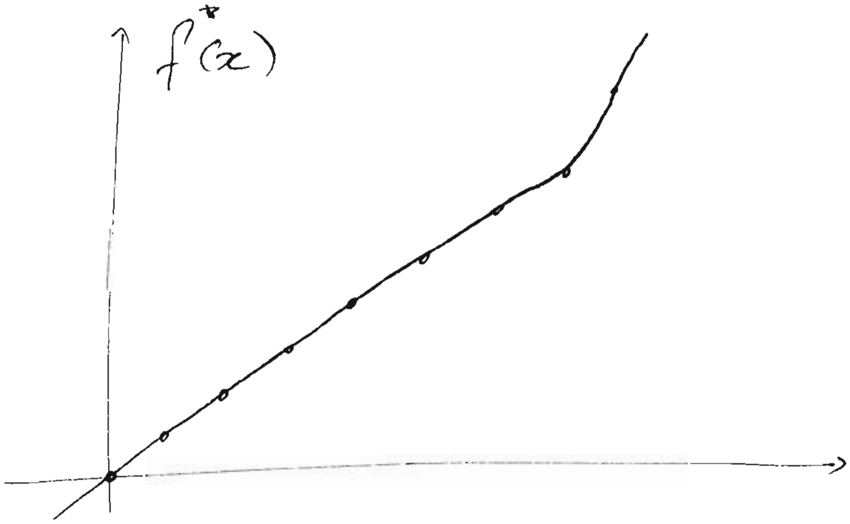


Thus P^* looks like



What we observe is that a <u>local</u> change in y-values caused a <u>global</u> (and drastic) change in $P(x)$.

Perhaps the "real" function f would have exhibited a more graceful and localized change, e.g.:



We will use the following theorem to compare the "real" function f being sampled, and the reconstructed interpolant $p(x)$

<u>Theorem 1</u> Let:

- $x_1 < x_2 < \dots < x_{n-1} < x_n$
- $y_k = f(x_k)$ $k = 1, 2, \dots, n$, where f is a function which is n-times differentiable with continuous derivatives
- $P(x)$ is a polynomial that interpolates $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

<u>Then</u> for any $x \in (x_1, x_n)$ there exists a $\theta = \theta(x) \in (x_1, x_n)$ such that

$$f(x) - P(x) = \frac{f^{(n)}(\theta)}{n!}(x - x_1)(x - x_2) \cdots (x - x_n)$$

This theorem may be difficult to apply directly, since:

→ θ is not known

→ θ changes with x

→ The n-th derivative $f^{(n)}(x)$ may not be fully known.

However, we can use it to derive a conservative bound:

<u>Theorem 2</u> If $M = \max_{x \in [x_1, x_n]} |f^{(n)}(x)|$

and $h = \max_{1 \le i \le n-1} |x_{i+1} - x_i|$

then $|f(x) - p(x)| \le \dfrac{M h^n}{4n}$ for all $x \in [x_1, x_n]$.

How good is this, especially when we keep adding more and more data points (e.g. $n \to \infty$ and $h \to 0$)

This really depends on the higher order derivatives of $f(x)$ ... For example

$f(x) = \sin x \quad x \in [0, 2\pi]$

All derivatives of f are $\pm \sin x$ or $\pm \cos x$

thus $|f^{(k)}(x)| \le 1$ for <u>any</u> k

In this case $M = 1$, and as we add more (and denser) data points we have $|f(x) - p(x)| \le \dfrac{Mh^n}{4n} \xrightarrow[n \to \infty]{h \to 0} 0$

For some functions, however, the values of $|f^{(k)}(x)|$ grow vastly as $k \to \infty$ (i.e. when we introduce additional points). e.g.:

$$f(x) = \frac{1}{x} \Rightarrow |f^{(n)}(x)| = n! \frac{1}{x^{n+1}}$$

$x \in (0.5, 1)$

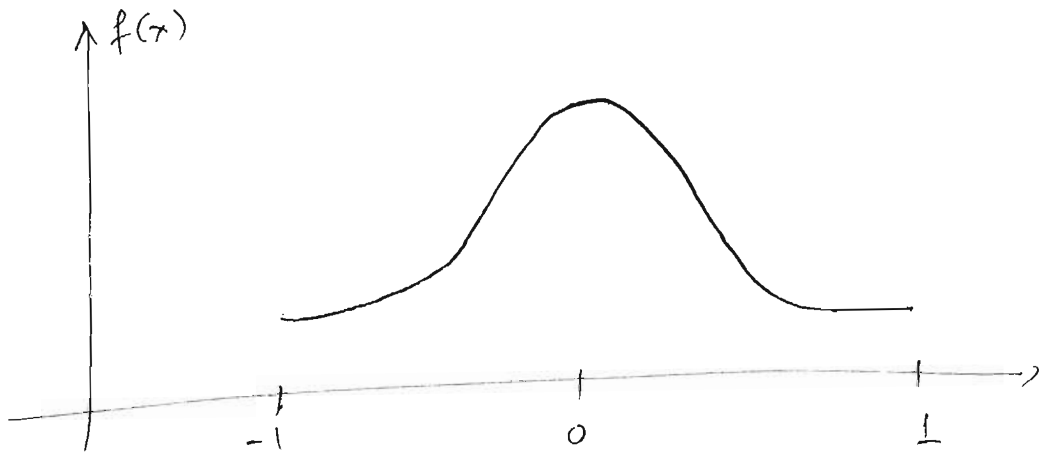
$$M_n = \max_{x \in (0.5, 1)} |f^{(n)}(x)| = n! \cdot 2^n$$

in this case, as $n \to \infty$:

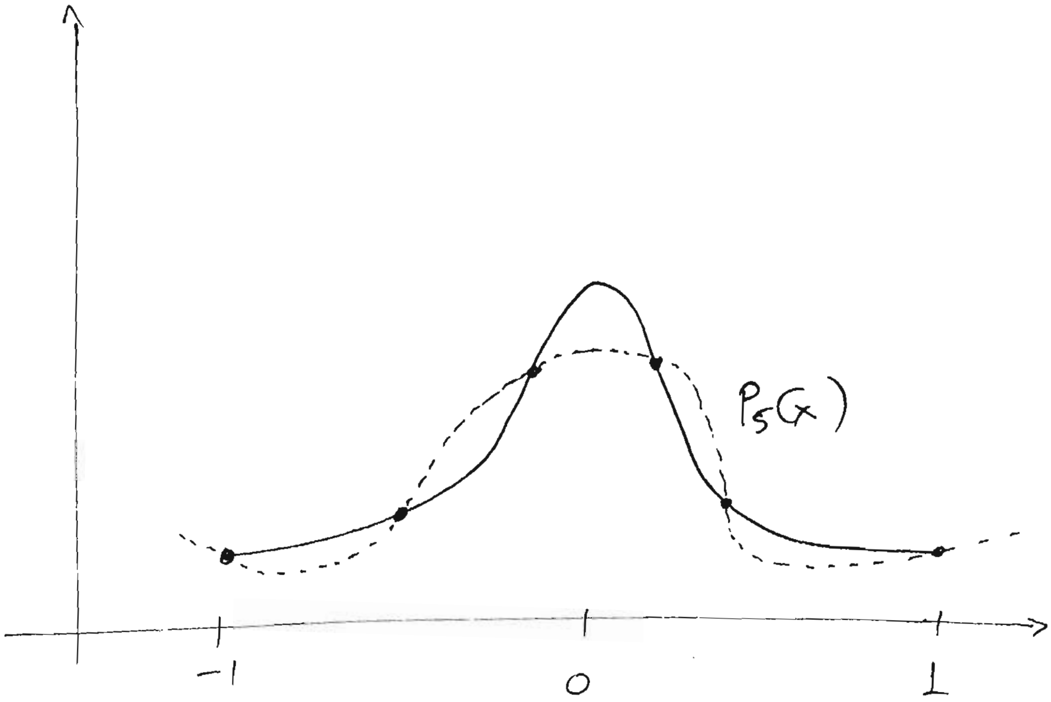
$$\frac{M_n h^n}{4n} = \frac{n!\, 2^n h^n}{4n} \xrightarrow[n \to \infty]{} +\infty \quad !$$

Another commonly cited counter-example is Runge's function:

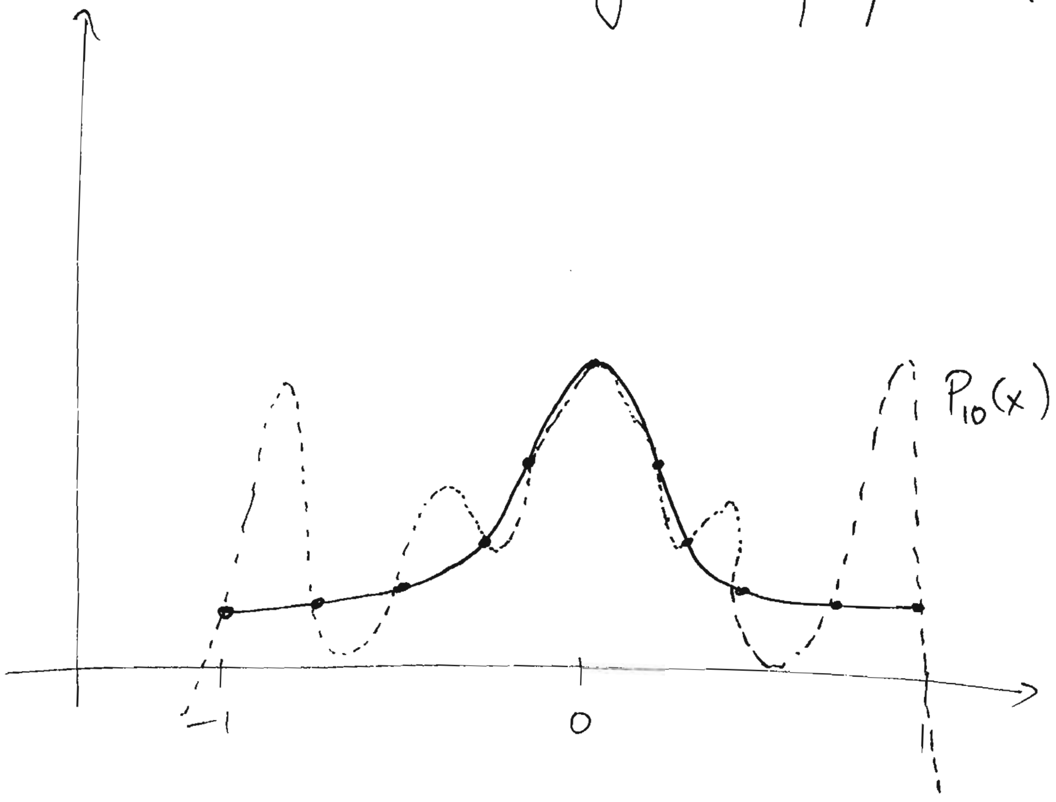
$$f(x) = \frac{1}{1 + 25x^2}$$



Approximation with a degree=5 polynomial



Approximation with a degree=10 polynomial



Thus in this case the polynomials $P_k(x)$ do <u>not</u> uniformly converge to $f(x)$ as we add more points

A possible improvement stems from the following idea:

$$f(x) - P(x) = \frac{f^{(n)}(\theta)}{n!} \underbrace{(x-x_1)\cdots(x-x_n)}$$

This we have no control over

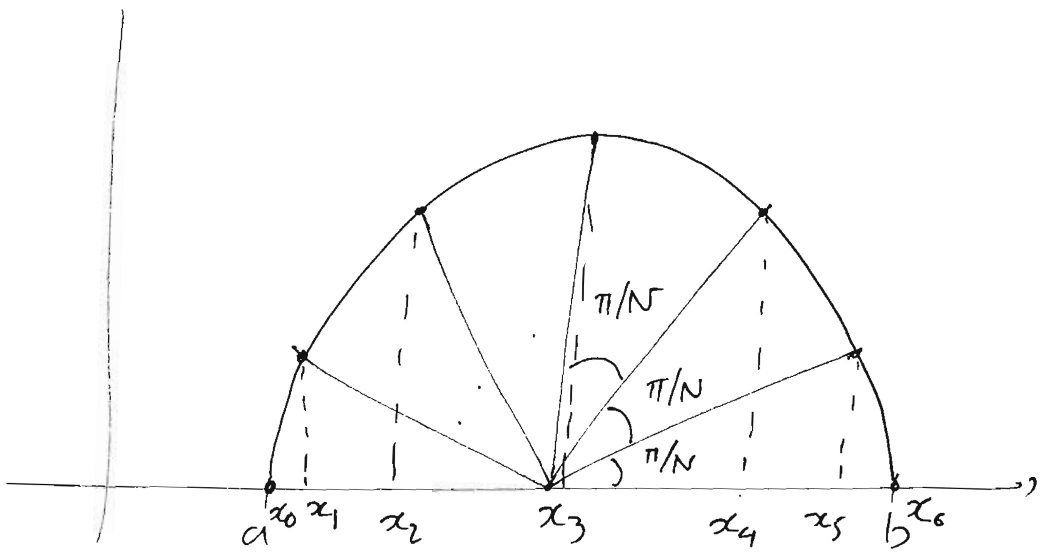
Idea: Select the points $x_1, x_2, \ldots, x_n$ to minimize this product.

The value of the product $(x-x_1)\cdots(x-x_n)$ is minimized by selecting the x_i's as the <u>Chebyshev points</u>.

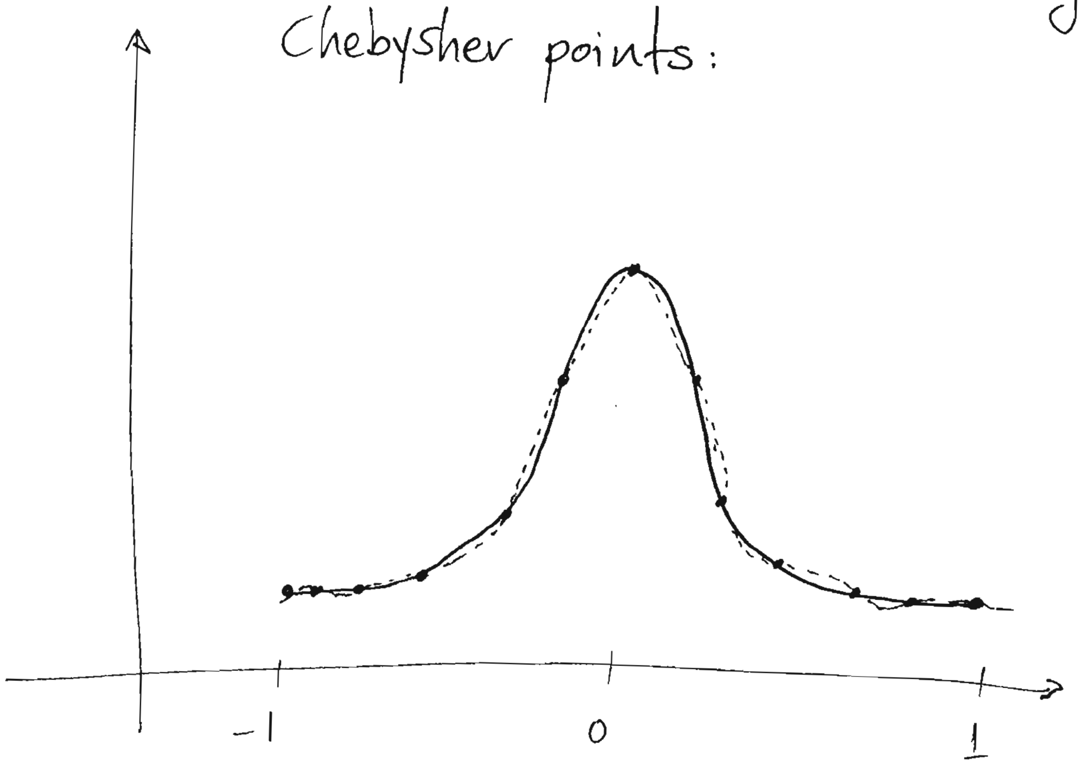
If the interpolation interval is $[a,b]$, the Chebyshev points are given by:

$$x_i = a + (b-a)\sin\left(\frac{i\pi}{N}\right) \qquad i = 0,1,2,\ldots,N$$

Graphically, these points are the projections on the x-axis of $(N+1)$ points located along the half circle with diameter the interval $[a,b]$, at equal arc-lengths:



Now, we can re-try Runge's function using Chebyshev points:



In fact, it is possible to show that, using Chebyshev points, we can guarrantee that

$$|f(x)-P(x)| \xrightarrow[n\to\infty]{} 0$$

provided that over $[a,b]$ both $f(x)$ and its derivative $f'(x)$ remain bounded (The benefit is that this condition does not place restrictions on higher-order derivatives of $f(x)$)