Theorem: If an integration rule integrates exactly any polynomial up to degree \((d-1)\), then the global error is \(O(h^d)\) or better, i.e. the rule is at least \(d\)-order accurate.

Methodology:

* Test the integration rule on monomials of degree 0, 1, 2, ..., i.e. on \(f(x) = 1\), \(f(x) = x\), \(f(x) = x^2\), ...

* If \(f(x) = x^d\) is the \(d+1\)st test function that is not integrated exactly, the order of accuracy is equal to \(d\).

Example: Trapezoidal rule

\[
I = \int_{a}^{b} f(x) \, dx \approx \frac{f(a) + f(b)}{2} (b-a)
\]

\(f(x) = 1\) : \(I_{\text{Trapez}} = \frac{b + a}{2} (b-a) = b - a \quad (= \text{exact})\)

\(f(x) = x\) : \(I_{\text{Trapez}} = \frac{a + b}{2} (b-a) = b^2 - a^2 \quad (= \text{exact})\)

\(f(x) = x^2\) : \(I_{\text{Trapez}} = \frac{a^2 + b^2}{2} (b-a) \quad \text{not exact} \quad \left(\int_{a}^{b} x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}\right)\)

Thus, rule is \(2\)-nd order accurate.
Initial value problems for 1st order differential equations

In this last part of our class we will turn our attention to differential equation problems, of the form:

Find the function \( y(t) : [t_0, +\infty) \to \mathbb{R} \) that satisfies the "ordinary differential equation (ODE)":

\[
y'(t) = f(t, y(t)) \quad \text{ (for a certain function } f) \]

and \( y(t_0) = y_0 \) \( \quad \text{(This is called an initial value problem)} \)

\[\text{Example}\]

The velocity \( v(t) \) of a vehicle over the time interval \([0, 5]\) satisfies \( v(t) = t(t+1) \). At time \( t=0 \), the vehicle starts from position \( x(0) = 5 \). What is \( x(t) \), \( t \in [0, 5] \)?

\[\text{Ans: Given by IVP } \quad x'(t) = t(t+1) \]

\[x(0) = 5 \]

\[\text{(Since } x'(t) = v(t)) \]
The concentration \( y(t) \) of a chemical species is given by:

\[
y'(t) = y(t^2 + t) \\
y(0) = 1
\]

(Here \( f(t, y) = y(t^2 + t) \)).

Of course in certain cases we can solve this differential equation exactly, e.g. in the last example:

\[
y'(t) = y(t^2 + t) \Rightarrow \frac{y'}{y} = t^2 + t
\]

\[
\Rightarrow \int_{t_0}^{t} \frac{y'}{y} \, dt = \int_{t_0}^{t} (t^2 + t) \, dt \Rightarrow
\]

\[
[\ln y]_{t_0}^{t} = \left[ \frac{t^3}{3} + \frac{t^2}{2} \right]_{t_0}^{t} \Rightarrow \ln y(t) - \ln y(0) = \frac{t^3}{3} + t
\]

\[
\Rightarrow y(t) = e^{\frac{t^3}{3} + t}
\]
However we do not want to depend in our ability to solve the O.D.E. exactly, since:

→ An exact solution may not be analytically expressible in closed form
→ The exact solution may be too complicated and (very important:)

→ The function \( f(t, y) \) may not be available as a formula; e.g., it could result from a black-box computer program.

**Solution:** APPROXIMATE the solution to the differential equation

**General methodology ("1-step methods")**

- Consider discrete points in time
  \[ t_0 < t_1 < t_2 < \ldots < t_k < \ldots \]
  If we set \( \Delta t_k = t_{k+1} - t_k \) and \( \Delta t_k = \Delta t = \text{const.} \)
  then \( t_k = t_0 + k \cdot \Delta t. \)
- Use the notation \( y_k = y(t_k) \).
Use the values $t_k, y_k$ and the ODE $y'(t) = f(t, y)$ to approximate $y_{k+1}$.

**Method:**

\[
y'(t) = f(t, y)
\]

\[
\int_{t_k}^{t_{k+1}} y'(t) \, dt = \int_{t_n}^{t_{k+1}} f(t, y) \, dt
\]

\[
\Rightarrow y(t_{k+1}) - y(t_n) = \int_{t_n}^{t_{k+1}} f(t, y) \, dt
\]

Thus:

\[
y_0 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \ldots
\]
For example, if we approximate this integral with the rectangle rule \[ \int_{a}^{b} f(x) \, dx \approx f(a)(b-a) \], we get

\[ y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(t,y) \, dt \approx f(t_{k+1}, y_{k+1})(t_{k+1} - t_k) \]

\[ \Rightarrow y_{k+1} = y_k + \Delta t f(t_{k+1}, y_{k+1}) \]

Forward Euler method, or Euler's method, or Explicit Euler's method.

Easy to evaluate: Plug in \( t_k, y_k \) \( \Rightarrow \) obtain \( y_{k+1} \).

Now if we had used the "right-sided" rectangle rule \[ \int_{a}^{b} f(x) \, dx \approx f(b)(b-a) \], we would obtain:

\[ y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(t,y) \, dt \approx f(t_{k+1}, y_{k+1}) \Delta t \]

\[ \Rightarrow y_{k+1} = y_k + \Delta t f(t_{k+1}, y_{k+1}) \]

Backward Euler method, or Implicit Euler method.

\( \text{Note: We need to solve a (possibly nonlinear) equation to obtain } y_{k+1} \) \( \text{(} y_{k+1} \text{ is not isolated in this equation)} \).
One more variant: \textbf{trapezoidal rule}

\[ \int_a^b f(x) \, dx = \frac{f(a) + f(b)}{2} \cdot (b-a) \]

\[ y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(t, y) \, dt \approx \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2} \cdot \Delta t \]

\[ \implies y_{k+1} = y_k + \frac{\Delta t}{2} \left\{ f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \right\} \]

\text{example:} \quad \dot{y}(t) = -ty^2 \quad \text{using trap. rule}

\[ y_{k+1} = y_k + \frac{\Delta t}{2} \left\{ -t_k y_k^2 - t_{k+1} y_{k+1}^2 \right\} \]

Let: \quad t_k = 0.9 \quad y_k = 1 \quad \Delta t = 0.1 \quad y_{k+1} = \frac{1}{1 + 0.05 \left\{ -0.9 - 1 \cdot y_{k+1}^2 \right\}}

\[ \Rightarrow 0.05y_{k+1}^2 + y_{k+1} + 1.045 = 0 \quad \Rightarrow \text{solve quadratic to get} \quad y_{k+1} \]
Another example:

\[ y'(t) = -2 y(t) \]
\[ y(0) = 1 \]

\{ exact \ solution \ \ \ y(t) = e^{-2t} \}

Using Forward Euler:

\[ y_{k+1} = y_k + \Delta t \cdot f(t_k, y_k) \]
\[ = y_k - 2\Delta t \cdot y_k = (1 - 2\Delta t) y_k \]

Thus

\[ y_1 = (1 - 2\Delta t) y_0 \]
\[ y_2 = (1 - 2\Delta t) y_1 = (1 - 2\Delta t)^2 y_0 \]
\[ \vdots \]
\[ y_n = (1 - 2\Delta t)^n y_0 \]

How does this behave when \( \Delta t \to 0 \)?

\[ (1 - 2\Delta t)^n = \left[ \left( 1 + \frac{1}{-2\Delta t} \right)^{-\frac{1}{2\Delta t}} \right]^{-2\Delta t} \]

Using \( \lim_{\Delta t \to 0} \left( 1 + \frac{1}{x} \right)^x = e \), \( \lim_{\Delta t \to 0} (1 - 2\Delta t)^k = e^{-2k\Delta t} - 2\Delta t \)

Thus, when \( \Delta t \to 0 \)

\[ y_n \to e^{-2tn} \quad (\text{compare with exact solution } y(t) = e^{-2t}) \]