Assessing order of accuracy in integration rules

<u>Theorem</u> If an integration rule integrates <u>exactly</u> any polynomial up to degree $(d-1)$, then the global error is $O(h^d)$ or better, i.e. the rule is at least d-order accurate.

Methodology

* Test the integration rule on <u>monomials</u> of degree $0, 1, 2, \ldots$, i.e. on $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, ...
* If $f(x) = x^d$ is the 1st test function that is <u>not</u> integrated exactly, the order of accuracy is equal to $\underline{\underline{d}}$

<u>Example</u> Trapezoidal rule $\quad I = \int_a^b f(x)\,dx \approx \dfrac{f(a)+f(b)}{2}(b-a)$

$f(x) = 1$: $I_{trap} = \dfrac{1+1}{2} \cdot (b-a) = b - a \quad (= \text{exact})$

$f(x) = x$: $I_{trap} = \dfrac{a+b}{2}(b-a) = \dfrac{b^2}{2} - \dfrac{a^2}{2} \quad (= \text{exact})$

$f(x) = x^2$ $\quad I_{trap} = \dfrac{a^2+b^2}{2}(b-a)$ $\underline{\text{not exact}}$ $\quad \left(\int_a^b x^2 dx = \dfrac{b^3}{3} - \dfrac{a^3}{3} \right)$

Thus, rule is <u>2nd</u> order accurate.

Initial value problems for 1st order differential equations

In this last part of our class we will turn our attention to differential equation problems, of the form:

Find the function $y(t) : [t_0, +\infty) \to \mathbb{R}$
that satisfies the "ordinary differential equation (ODE)":

$$y'(t) = f(t, y(t)) \quad \text{(for a certain function } f\text{)}$$

and $y(t_0) = y_0$ (This is called an initial value problem)

example

→ The velocity of a vehicle over the time interval $[0,5]$ satisfies $v(t) = t(t+1)$. At time $t=0$, the vehicle starts from position $x(0) = 5$. What is $x(t)$, $t \in [0,5]$?

Ans Given by IVP

$$x'(t) = t(t+1)$$
$$x(0) = 5$$

(Since $x'(t) = v(t)$)

<u>ex 2</u>: The concentration $y(t)$ of a chemical species is given by:

$$y'(t) = y\,(t^2+1)$$

$$y(0) = 1 \qquad \left(\text{Here } f(t,y) = y\,(t^2+1)\right).$$

Of course in certain cases we can solve this differential equation <u>exactly</u>, e.g. in the last example:

$$y'(t) = y\,(t^2+1) \Rightarrow \frac{y'}{y} = t^2+1$$

$$\Rightarrow \int_{t_0=0}^{t} \frac{y'}{y}\,d\tau = \int_{t_0=0}^{t} (\tau^2+1)\,d\tau \Rightarrow$$

$$\Rightarrow [\ln y]_{t_0=0}^{t} = \left[\frac{\tau^3}{3}+\tau\right]_0^t \Rightarrow \ln y(t) - \underbrace{\cancel{\ln y(0)}}_{=0} = \frac{t^3}{3}+t$$

$$\Rightarrow y(t) = e^{\left(\frac{t^3}{3}+t\right)}$$

However we do not want to depend in our ability to solve the O.D.E. exactly, since:

→ An exact solution may not be analytically expressible in closed form

→ The exact solution may be too complicated and (very important:)

→ The function $f(t,y)$ may not be available as a formula; e.g. it could result from a black-box computer program.

Solution: APPROXIMATE the solution to the differential equation

General methodology ("1-step methods")

- Consider discrete points in time

  $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$

  If we set $\Delta t_k = t_{k+1} - t_k$ and $\Delta t_k = \Delta t = \text{const}$,

  then $t_k = t_0 + k \cdot \Delta t$.

- Use the notation $y_k = y(t_k)$.

- Use the values t_k, y_k AND the ODE.

$y'(t) = f(t, y)$ to approximate y_{k+1}.

<u>Method</u>:

$$y'(t) = f(t,y)$$

$$\Rightarrow \int_{t_k}^{t_{k+1}} y'(\tau)\,d\tau = \int_{t_k}^{t_{k+1}} f(\tau, y)\,d\tau$$

$$\Rightarrow \underbrace{y(t_{k+1})}_{:= y_{k+1}} - \underbrace{y(t_k)}_{:= y_k} = \underbrace{\int_{t_k}^{t_{k+1}} f(\tau, y)\,d\tau}_{\text{Approximate using integration rule!}}$$

Thus

$$\begin{array}{ccccccc} y_0 & \longrightarrow & y_1 & \longrightarrow & y_2 & \longrightarrow & y_3 \cdots \\ t_0 & & t_1 & & t_2 & & \end{array}$$

For example, if we approximate this integral with the rectangle rule $\int_a^b f\,dx \cong f(a)(b-a)$, we get

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$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y)\,d\tau \approx f(t, y)(t_{k+1} - t_k)$$

$$\Rightarrow \boxed{y_{k+1} = y_k + \Delta t f(t_k, y_k)}$$

Forward Euler method, or Euler's method, or Explicit Euler's method

Easy to evaluate: Plug in $t_n, y_k \longrightarrow$ obtain y_{n+1}.

Now if we had used the "right-sided" rectangle rule

$\int_a^b f\,dx \approx f(b)(b-a)$, we would obtain:

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y)\,d\tau \approx f(t_{k+1}, y_{k+1})\,\Delta t$$

$$\Rightarrow \boxed{y_{k+1} = y_k + \Delta t\, f(t_{k+1}, y_{k+1})}$$

Backward Euler method, or Implicit Euler method.

Note: We need to solve a (possibly nonlinear) equation to obtain y_{k+1} (y_{n+1} is not isolated in this equation).

One more variant: trapezoidal rule 4/19/11

$$\int_a^b f\,dx = \frac{f(a)+f(b)}{2}\cdot(b-a)$$

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y)\,d\tau \approx \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2} \cdot \Delta t$$

$$\Rightarrow \boxed{y_{k+1} = y_k + \frac{\Delta t}{2}\left\{ f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \right\}}$$

example: $y'(t) = -ty^2$ using trap. rule

$$y_{k+1} = y_k + \frac{\Delta t}{2}\left\{ -t_k y_k^2 - t_{k+1} y_{k+1}^2 \right\}$$

Let: $t_k = 0.9$ $y_k = 1$

$\Delta t = 0.1$

$$y_{k+1} = 1 + 0.05\left\{ -0.9 - 1 \cdot y_{k+1}^2 \right\}$$

$\Rightarrow 0.05 y_{k+1}^2 + y_{k+1} + 1.045 = 0 \Rightarrow$ solve quadratic to get y_{k+1}

Another example:

$$\left.\begin{array}{l} y'(t) = -2y(t) \\ y(0) = 1 \end{array}\right\} \quad \text{exact solution } y(t) = e^{-2t}$$

Using Forward Euler:

$$y_{k+1} = y_k + \Delta t\, f(t_k, y_k)$$

$$= y_k - 2\Delta t\, y_n = (1 - 2\Delta t)\, y_k$$

Thus

$$y_1 = (1-2\Delta t)\, y_0$$

$$y_2 = (1-2\Delta t)\, y_1 = (1-2\Delta t)^2\, y_0$$

$$\vdots$$

$$y_n = (1-2\Delta t)^k\, y_0$$

How does this behave when $\Delta t \to 0$?

$$(1-2\Delta t)^k = \left[\left(1 + \frac{1}{-\frac{1}{2\Delta t}}\right)^{-\frac{1}{2\Delta t}}\right]^{-2k\Delta t}$$

Using $\lim \left(1 + \frac{1}{x}\right)^x = e \Rightarrow \lim\limits_{\Delta t \to 0} (1-2\Delta t)^k = e^{-2k\Delta t} = e^{-2t_n}$

Thus, when $\Delta t \to 0$ $y_n \to e^{-2t_n}$ (compare with exact solution $y(t) = e^{-2t}$).