Initial value problems for 1st order ODE's

Problem statement: Find $y(t)$: $[t_0, +\infty)$

s.t

$$y'(t) = f(t,y)$$
$$y(t_0) = y_0$$

Method (1-step algorithms)

- Set $t_k = t_0 + k \cdot \Delta t$
- Define $y_k := y(t_k)$
- Iteratively approximate

$$y_{k+1} = y_k + \Delta t \, f(t_k, y_k)$$
(Forward Euler)

Explicit

$$y_{k+1} = y_k + \Delta t \, (t_{k+1}, y_{k+1})$$
(Backward Euler)

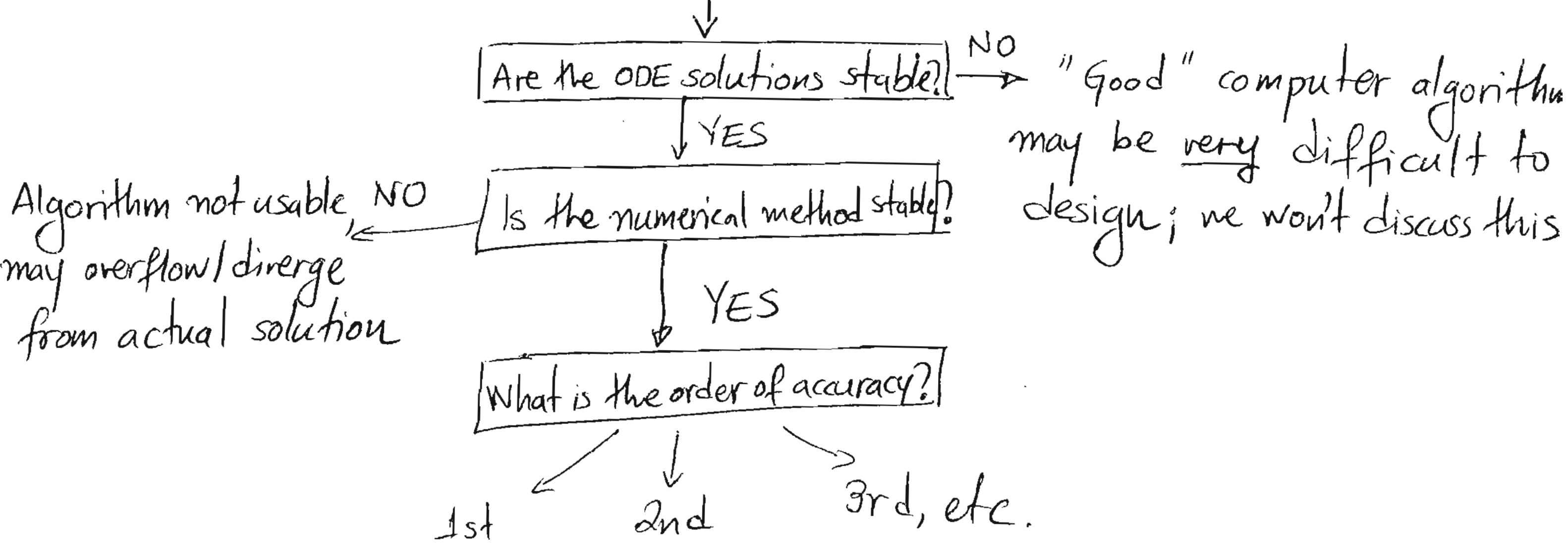
$$y_{k+1} = y_k + \frac{\Delta t}{2}\left[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right]$$
(Trapezoidal)

Implicit (need to solve an equation for y_{k+1}).

Before we use one of these algorithms in practice we need to examine their limitations, and ensure they are usable for a specific problem. We look at the following

Properties of the ODE itself	Properties of the numerical method
→ Are the <u>solutions</u> to the ODE <u>stable</u>?	→ Is the numerical method stable? Under what conditions?
	→ What is the accuracy of the method?

Schematically, we have start



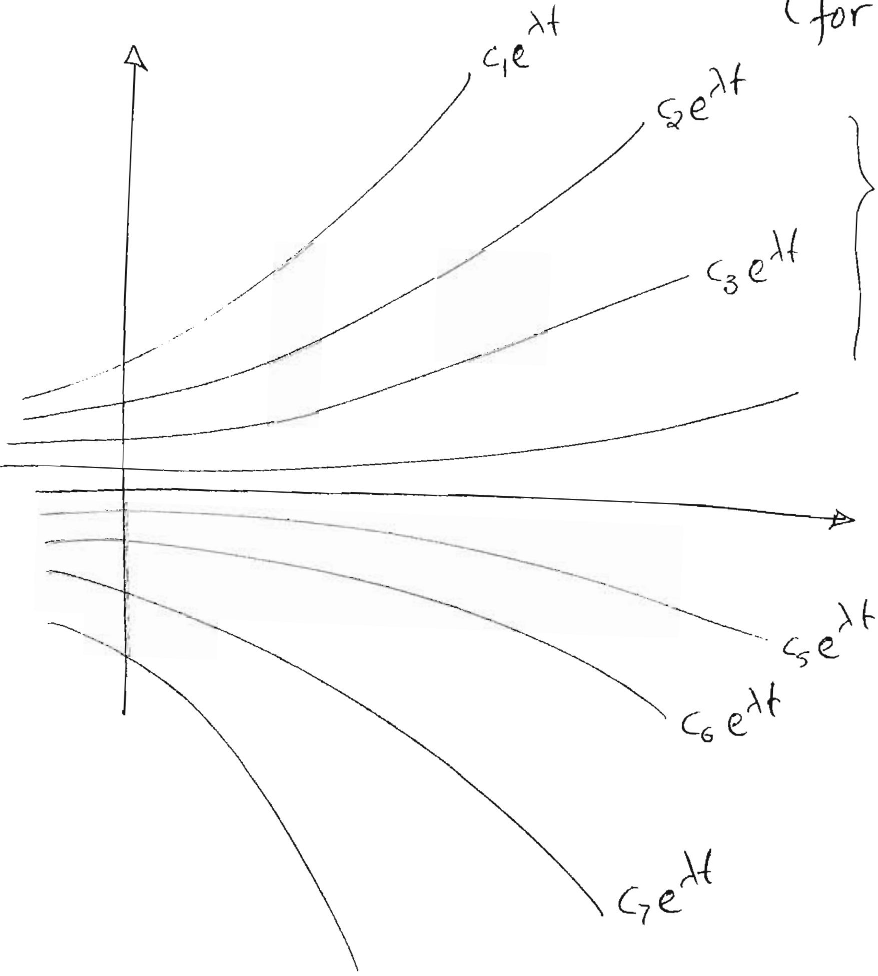
Stability of solutions to ODE

We formulated an IVP as:

$$y'(t) = f(t,y) \longleftarrow \text{ODE}$$
$$y(t_0) = y_0 \longleftarrow \text{initial condition}$$

Under normal circumstances we expect this to have a unique solution; however if we omit the initial condition, we get an entire <u>family</u> of solutions to the ODE.

<u>e.g</u> $y'(t) = \lambda y(t) \Rightarrow$ Exact solution $y(t) = ce^{\lambda t}$ (for any arbitrary $c \in \mathbb{R}$)



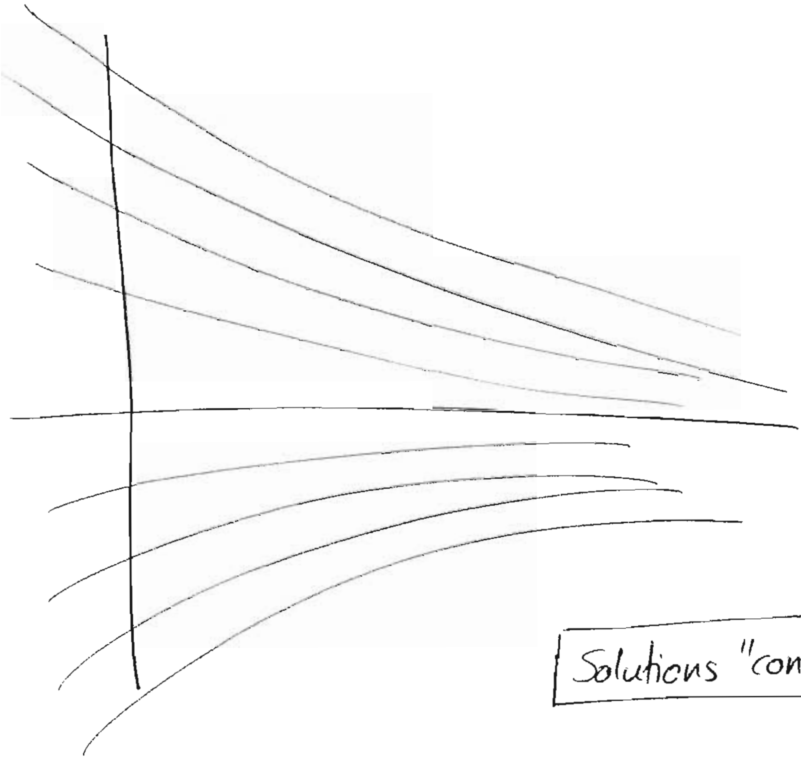
positive c_i's

Case $\lambda > 0$

$\boxed{\text{Solutions "diverge"}}$

Negative c_i's.

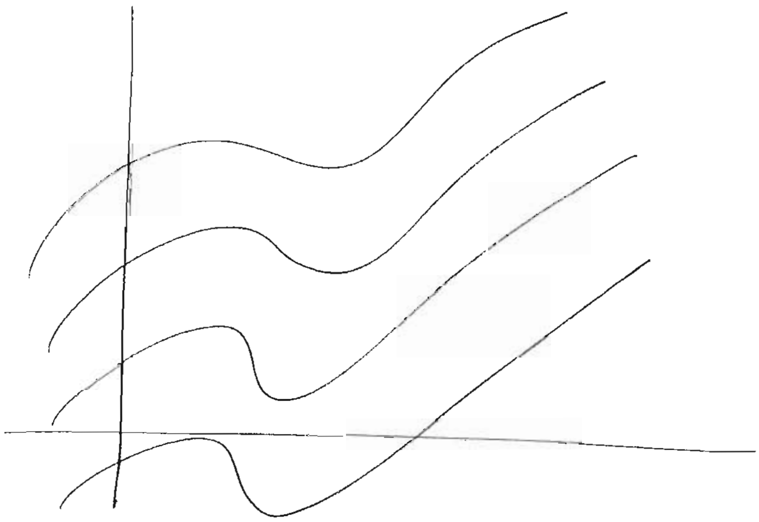
Case $\lambda < 0$



$\boxed{\text{Solutions "converge"}}$

- $y'(t) = f(t)$ (f is a function of t alone, not y)

Exact solution: $y(t) = \displaystyle\int_{t_0}^{t} f(\tau)\,d\tau + C \quad , \quad C \in \mathbb{R}$



$\boxed{\text{Solutions stay at fixed distance apart}}$

Definition

- An ODE is said to have stable solutions if the distance between any 2 solutions y & $\hat{y}$ remains bounded, i.e $|y(t)-\hat{y}(t)| \leq \text{const} \quad \forall t \geq t_0$

  (Strictly speaking, we must also be able to make this constant arbitrarily small, by bringing the initial values y_0 & $\hat{y}_0$ closer together)

- If we additionally have that for any 2 solutions $y(t)$ & $\hat{y}(t)$ we have $\lim_{t\to\infty} |y(t)-\hat{y}(t)| = 0$, the ODE has asymptotically stable solutions.

  Note: If the ODE is asymptotically stable then, it is stable, too.

- Otherwise (i.e. when solutions diverge away from one another) the ODE is said to have unstable solutions.

The ODE $\boxed{y'(t) = \lambda y(t)}$ is called the 4/21/11 LG

model 1st order ODE and is extremely useful as an example in the analysis of stability, etc. We have:

- When $\lambda < 0$ the solutions to the model ODE are asymptotically stable (converge towards one another)
- When $\lambda > 0$ the solutions are unstable (diverge away)
- When $\lambda = 0$ the solutions are stable although not asymptotically stable (they stay within bounded distance).

For a more general ODE $y' = f(t,y)$ the criteria are:

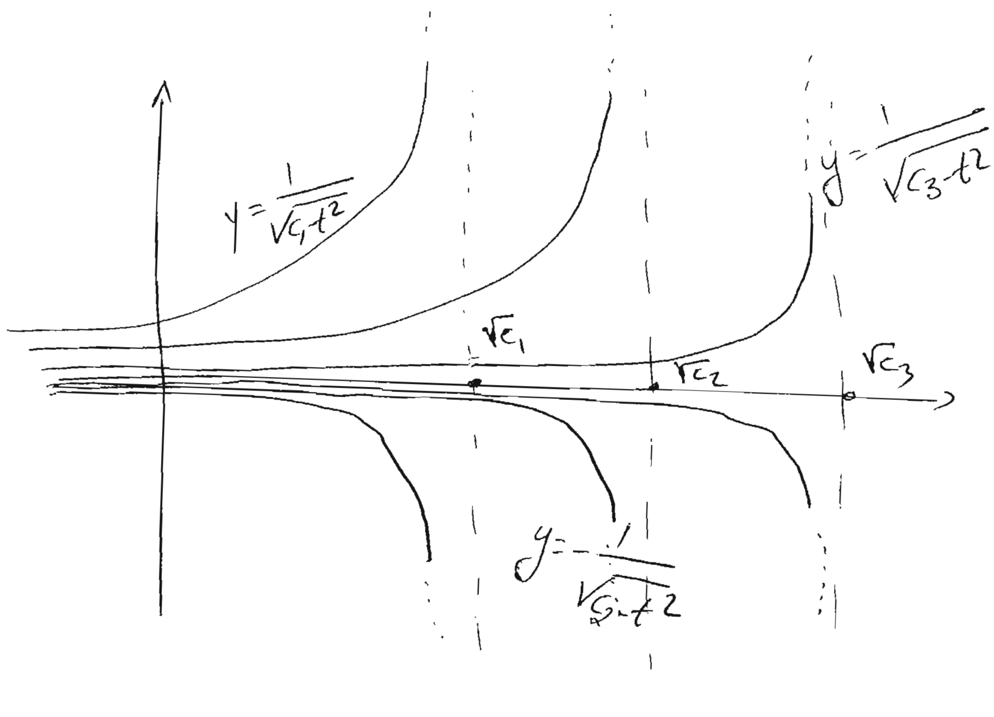
- If $\frac{\partial f}{\partial y}(t,y) < 0$ for all t & y, the solutions are asymptotically stable
- If $\frac{\partial f}{\partial y}(t,y) \leq 0$ $\forall t,y$, the solutions are stable (but not necessarily asymptotically stable).
- If $\frac{\partial f}{\partial y}(t,y)$ is positive or changes sign $\Rightarrow$ we cannot conclude stability with certainty.

Why do we ideally want ODEs with stable solutions?

→ Errors (approximation, truncation, roundoff) tend to move us away from the "intended" solution to an IVP, and onto another function from the family of solutions to the ODE. If the solution is stable (or even better, asymptotically stable) then the error remains bounded (or diminishes, for asymptotic stability) over time.

→ ODEs with unstable solutions are prone to developing problematic behaviors. For example, different solutions may become <u>undefined</u> after a certain (solution-dependent) point in time.

e.g. $y'(t) = ty^3 \Rightarrow$ Exact solution : $y(t) = \pm \frac{1}{\sqrt{c-t^2}}$



Designing a "usable" algorithm for approximating solutions to an unstable ODE is highly nontrivial, and we will not address it in CS412! So, we will continue under the premise that the ODE in question is stable.

→ Sometimes, even if the ODE is stable, an approximation method may diverge/overflow! e.g.

$y' = \lambda y \quad \lambda < 0 \quad$ (Exact solution $y(t) = y_0 e^{\lambda(t-t_0)}$)

Using Forward Euler:

$$y_{k+1} = y_k + \Delta t \, \lambda y_k = (1+\lambda\Delta t) y_k$$

$$\Rightarrow \quad y_k = (1+\lambda\Delta t)^k y_0$$

When $\lambda < 0$ the exact solution satisfies:

$$y(t) = y_0 e^{\lambda(t-t_0)} \xrightarrow[t\to\infty]{} 0$$

However, for the approximate solution

$$y_k = (1+\lambda\Delta t)^k y_0 \xrightarrow[t\to\infty]{} \begin{cases} \text{Converges to } 0, \text{ if } |1+\lambda\Delta t|<1 \\ \text{Diverges to } \pm\infty, \text{ if } |1+\lambda\Delta t|>1 \\ \text{Oscillates, if } |1+\lambda\Delta t| = 1. \end{cases}$$

Def A numerical method is called stable, when if applied to an ODE with stable solutions, exhibits the same asymptotic behavior with the exact solution when $t\to\infty$

In our case, the proper asymptotic behavior is $y_k \xrightarrow[t\to\infty]{} 0$, which is only guaranteed when

$$|1+\lambda\Delta t| < \quad \Leftrightarrow \quad -1 < 1+\lambda\Delta t < 1$$

$$\Leftrightarrow \quad -2 < \underbrace{\lambda\Delta t < 0}_{\substack{\text{always true} \\ \text{since } \lambda<0}} \quad \Leftrightarrow \quad -2 < -|\lambda|\Delta t$$

$$\Leftrightarrow \quad \boxed{\Delta t < \frac{2}{|\lambda|}}$$

Stability condition for Forward Euler!

(All implicit methods have some condition for stability)

What about Backward Euler? Again we test on the model stable ODE $y' = \lambda y$, $\lambda < 0$

$$y_{k+1} = y_k + \Delta t\, f(t_{k+1}, y_{k+1})$$

$$y_{k+1} = y_k + \Delta t \cdot \lambda\, y_{k+1}$$

$$(1 - \lambda \Delta t)\, y_{k+1} = y_k \Rightarrow y_{k+1} = \frac{1}{1 - \lambda \Delta t}\, y_k$$

$$\Rightarrow \boxed{y_k = \left(\frac{1}{1 - \lambda \Delta t}\right)^k y_0}$$

Here, in order to have $y_k \xrightarrow[k \to \infty]{} 0$, we need

$$\left|\frac{1}{1 - \lambda \Delta t}\right| < 1 \Leftrightarrow |1 - \lambda \Delta t| > 1$$ Always true! since $\lambda < 0$.

Thus, Backward Euler is unconditionally stable!

Similarly, for trapezoidal rule: 4/21/11 L11

(on model ODE $y' = \lambda y$, $\lambda < 0$)

$$y_{k+1} = y_k + \frac{\Delta t}{2}\left[f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \right]$$

$$\Rightarrow y_{k+1} = y_k + \frac{\Delta t}{2}\left[\lambda y_k + \lambda y_{k+1} \right]$$

$$\Rightarrow \left(1 - \frac{\lambda \Delta t}{2}\right) y_{k+1} = \left(1 + \frac{\lambda \Delta t}{2}\right) y_k$$

$$\Rightarrow \quad y_k = \left[\frac{\left(1 + \frac{\lambda \Delta t}{2}\right)}{\left(1 - \frac{\lambda \Delta t}{2}\right)} \right]^k y_0$$

For stability we need:

$$\left| \frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}} \right| < 1 \iff \left| 1 + \frac{\lambda \Delta t}{2} \right| < \underbrace{\left| 1 - \frac{\lambda \Delta t}{2} \right|}_{>0} \iff$$

$$\iff \left| 1 + \frac{\lambda \Delta t}{2} \right| < 1 - \frac{\lambda \Delta t}{2} \iff \underbrace{-1 + \frac{\lambda \Delta t}{2} < 1 + \frac{\lambda \Delta t}{2}}_{\text{Always true}} \underbrace{< 1 - \frac{\lambda \Delta t}{2}}_{\text{Always true, for } \lambda < 0}$$

Thus, the trapezoidal rule is also <u>unconditionally</u> stable!