Although using Chebyshev points mitigates some of the drawbacks of high-order polynomial interpolants, this is still a non-ideal solution, since:

- We do not always have the flexibility to pick the x_i's
- Polynomial interpolants of high degree typically require more than $O(n)$ computational cost to construct
- Local changes in the data points affect the entire extent of the interpolant.

A better remedy: Use piecewise polynomials

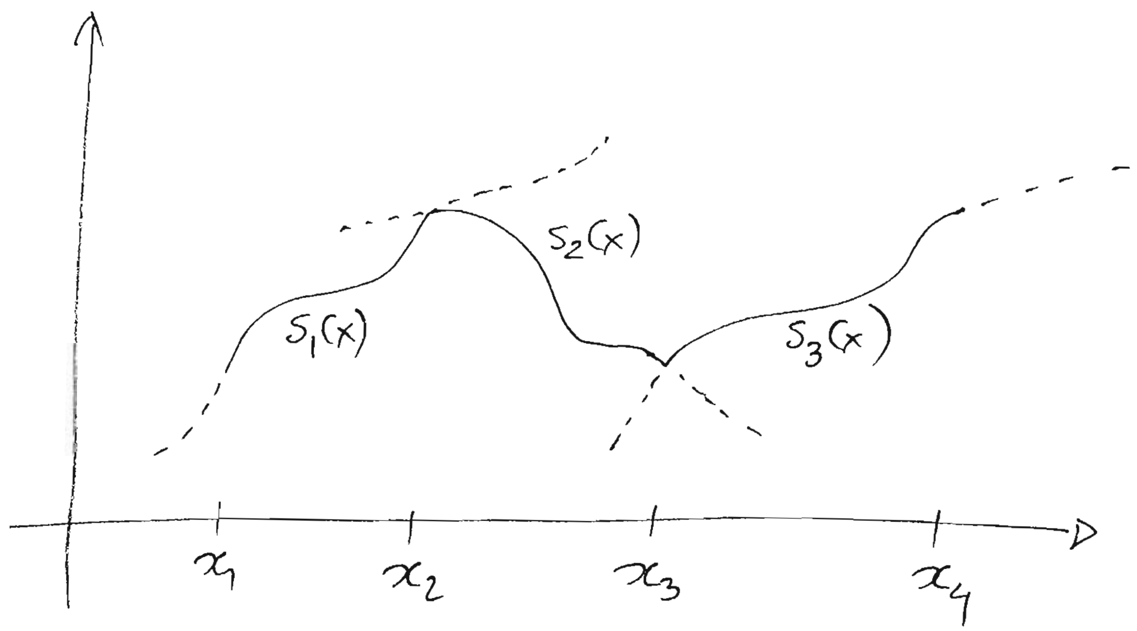
Assume that the x-values $\{x_i\}_{i=1}^{n}$ are sorted in ascending order:

$$a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Define
$$I_k = [x_k, x_{k+1}]$$
$$h_k = |x_{k+1} - x_k|$$

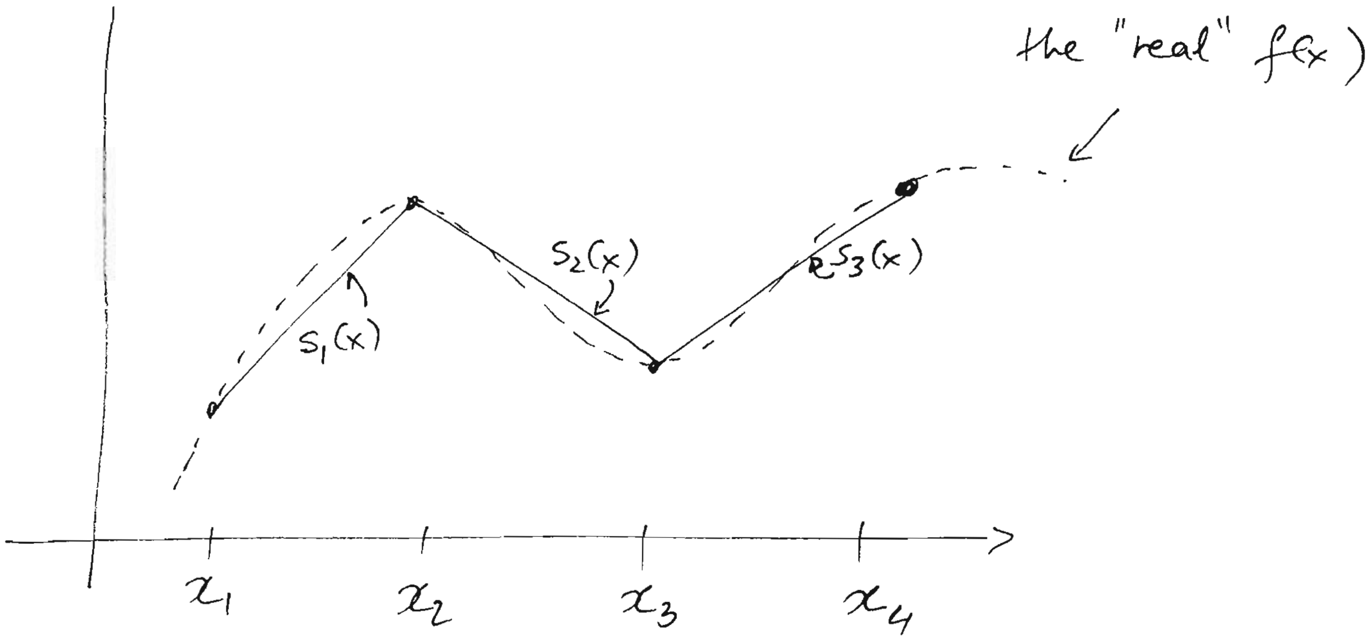
We also define the polynomials $s_1(x), s_2(x), \dots, s_{n-1}(x)$ and use each of them to define the interpolant $s(x)$ at the respective interval I_k:

$$s(x) = \begin{cases} s_1(x), & x \in I_1 \\ s_2(x), & x \in I_2 \\ \vdots \\ s_{n-1}(x), & x \in I_{n-1} \end{cases}$$



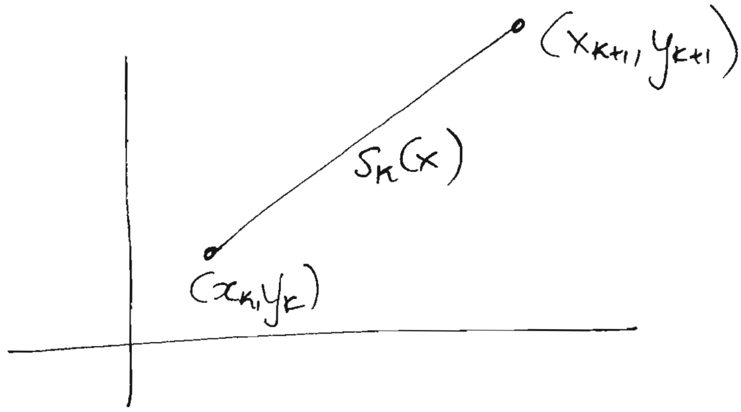
The benefit of using piecewise-polynomial interpolants, is that each $s_k(x)$ can be relatively low-order and thus non-oscillatory and easier to compute.

The simplest piecewise polynomial interpolant is a <u>piecewise linear</u> curve:



In this case every S_k can be written out explicitly as:

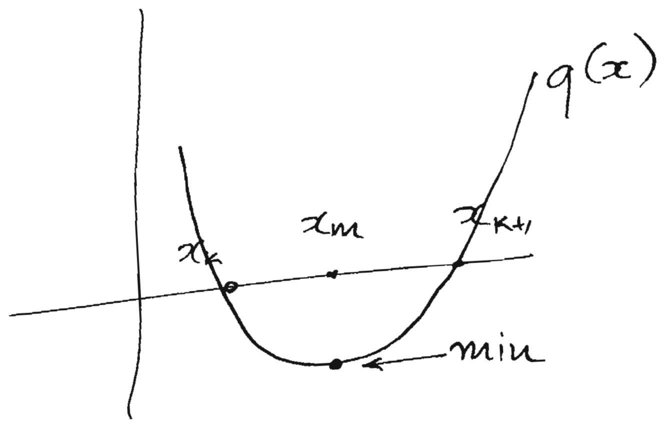
$$S_k(x) = y_k + \frac{y_{k+1} - y_k}{x_{k+1} - x_k}(x - x_k)$$



The next step is to examine the error $e(x) = f(x) - S_K(x)$ in the interval I_K. From the theorem we presented in the last lecture, we have that, for any $x \in I_K$ there is a $\theta_K = \theta(x_K)$ in I_K, such that:

$$e(x) = f(x) - S_K(x) = \frac{f''(\theta_K)}{2} \underbrace{(x-x_K)(x-x_{K+1})}_{q(x)} \qquad (1)$$

We are interested in the <u>maximum</u> value of $|q(x)|$ in order to determine a bound for the error. $q(x)$ is a quadratic function which crosses zero at x_K & x_{K+1}, thus the extreme value is obtained at the midpoint.



$$x_m = \frac{x_{K+1} + x_K}{2}$$

Thus $$|q(x)| \le |q(x_m)| = \left(\frac{x_{K+1} - x_K}{2}\right)^2 = \frac{h_K^2}{4}$$

for all $x \in I_K$.

Thus, using equation (1) we obtain:

$$|f(x)-S_K(x)| \le \max_{x\in I_K}\left|\frac{f''(x)}{2}\right| \cdot \max_{x\in I_K}\left|(x-x_K)\cdot(x-x_{K+1})\right|$$

$$= \max_{x\in I_K}\left|\frac{f''(x)}{2}\right| \cdot \frac{h_K^2}{4}$$

$$\Rightarrow \quad |f(x)-S_K(x)| \le \frac{1}{8}\max_{x\in I_K}|f''(x)|\cdot h_K^2$$

for all $x\in I_K$.

Additionally, if we assume all data points are equally spaced,

i.e $\quad h_1 = h_2 = \cdots = h_{n-1} = h \quad \left(= \frac{b-a}{n-1}\right)$

we can additionally write:

$$|f(x)-s(x)| \le \frac{1}{8}\max_{x\in[a,b]}|f''(x)|\cdot h^2$$

We often express the quantity on the right-hand side using the "infinity norm" of a given function, defined as

$$\|f\|_\infty = \max_{x\in[a,b]}|f(x)|$$

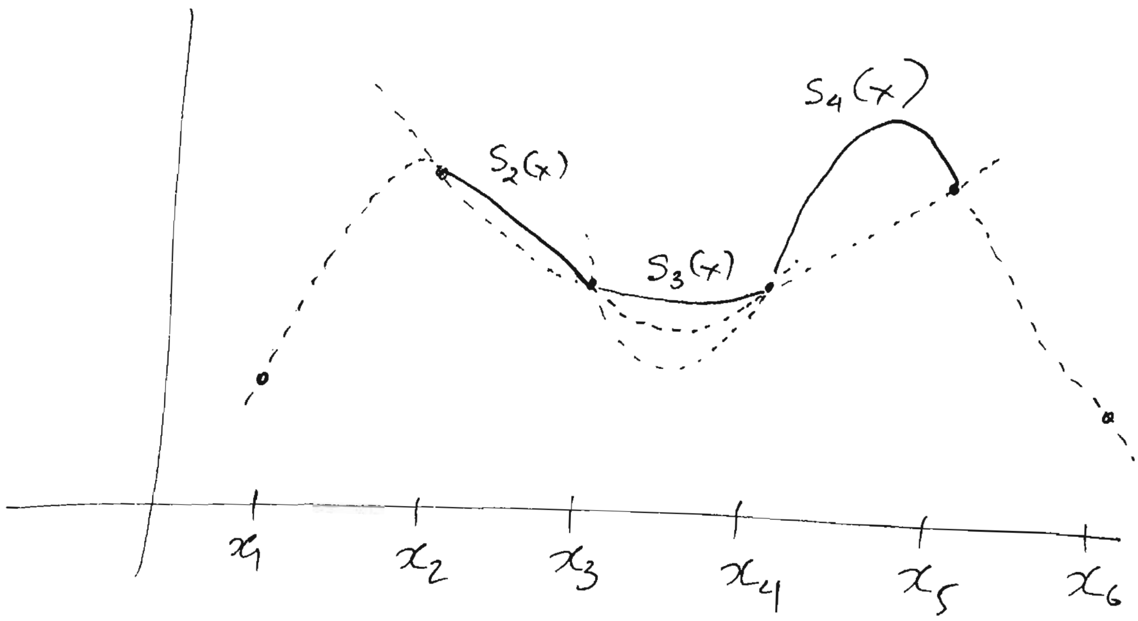
Thus, using this notation:

$$\boxed{|f(x)-s(x)| \leq \frac{1}{8} \|f''\|_\infty \cdot h^2}$$

Note that

- As $h \to 0$, the maximum discrepancy between f & s vanishes (proportionally to h^2)
- As we introduce more points, the quality of the approximation increases consistently, since the criterion above only considers the second derivative $f''(x)$ and not any higher order.

<u>A possible improvement</u> Piecewise cubic interpolation



In this approach each $s_k(x)$ is a cubic polynomial, designed such that it interpolates the 4 data points:

$$(x_{k-1}, y_{k-1}), (x_k, y_k), (x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2})$$

As we will see, the benefit is that the error can be made even smaller than with piecewise linear curves; the drawback is that (as seen in the last figure) $s(x)$ can develop "kinks" (or corners) where 2 pieces s_k & s_{k+1} are joined.

Error of piecewise cubics:

$$f(x) - s_k(x) = \frac{f''''(\theta_k)}{4!} \underbrace{(x-x_{k-1})(x-x_k)(x-x_{k+1})(x-x_{k+2})}_{q(x)}$$

An analysis similar to the linear case can show that $|q(x)| \le \frac{9}{16} \max\{h_{k-1}, h_k, h_{k+1}\}^4$

If we again assume $h_1 = h_2 = \cdots = h_k = h$, the error bound becomes:

$$|f(x)-s(x)| \leq \frac{1}{24} \|f''''\|_\infty \frac{9}{16} \cdot h^4$$

$$\Rightarrow \boxed{f(x)-s(x) \leq \frac{9}{384} \|f''''\|_\infty h^4}$$

The next possibility we shall consider, is a piecewise cubic curve

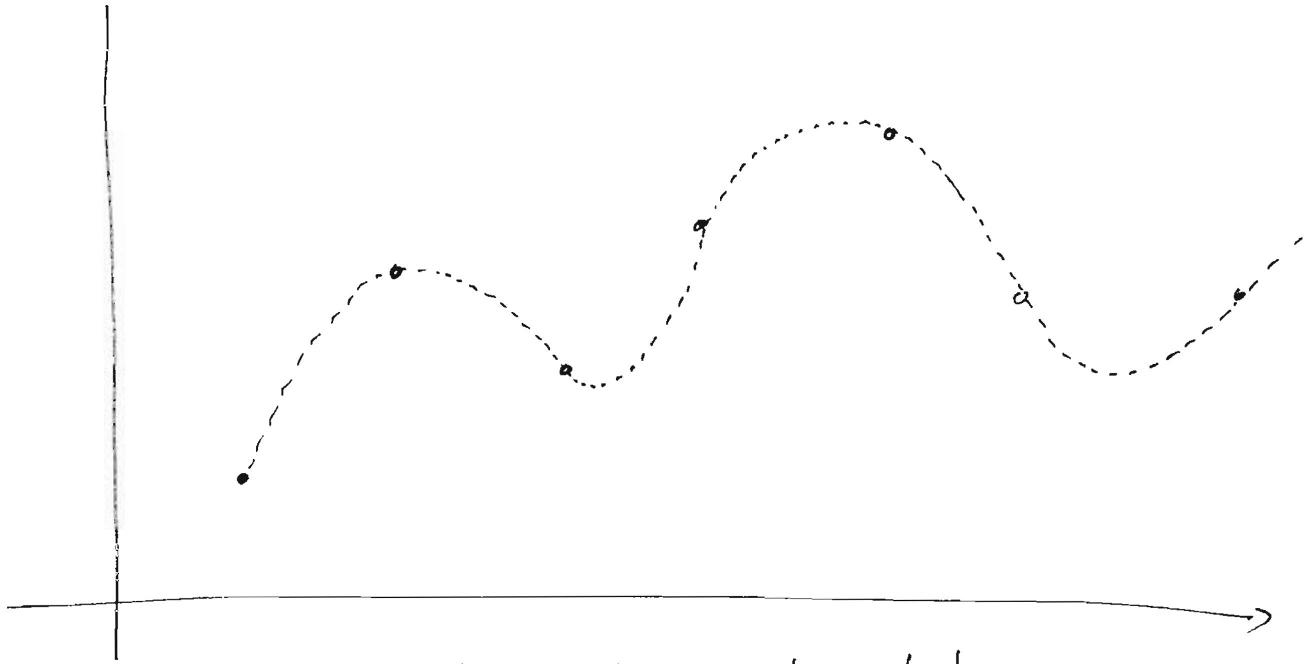
$$s(x) = \begin{cases} s_1(x) & x \in I_1 \\ \vdots & \\ s_{n-1}(x) & x \in I_{n-1} \end{cases}$$

where each $s_k(x) = a_3^{(k)} x^3 + a_2^{(k)} x^2 + a_1^{(k)} x + a_0^{(k)}$

and the coefficients $a_i^{(j)}$ are chosen as to <u>force</u> that the curve has continuous values, first and second derivatives:

$$s_k(x_{k+1}) = s_{k+1}(x_{k+1})$$
$$s_k'(x_{k+1}) = s_{k+1}'(x_{k+1})$$
$$s_k''(x_{k+1}) = s_{k+1}''(x_{k+1})$$

The curve constructed this way is called a <u>cubic spline</u> interpolant.



A cubic spline interpolation

Note the increased smoothness (continuity of values & derivatives) at the endpoints of each interval I_k.