Solving nonlinear equations (§2.4, 2.1)

In the previous lecture, we showed an iterative method for computing the square root $\sqrt{a}$, which only required elementary operations (e.g. addition, multiplication & division)

In summary, we had:

* Start by setting x_0 = < some initial guess for $\sqrt{a}$ >
* Generate the sequence $x_0, x_1, \dots, x_k, \dots$ by:

$$x_{k+1} = \frac{x_k^2 + a}{2x_k}$$

We also showed the following 2 facts:

* If the sequence $\{x_k\}$ converges, it will converge to a solution of $x^2 - a = 0$ (i.e. $x = \pm\sqrt{a}$)
* Assuming that for some k_0 we have that

$$\left|\frac{x_{k_0} - \sqrt{a}}{\sqrt{a}}\right| = \text{"small"} \quad \text{(for example, less than 1\%)}$$

then

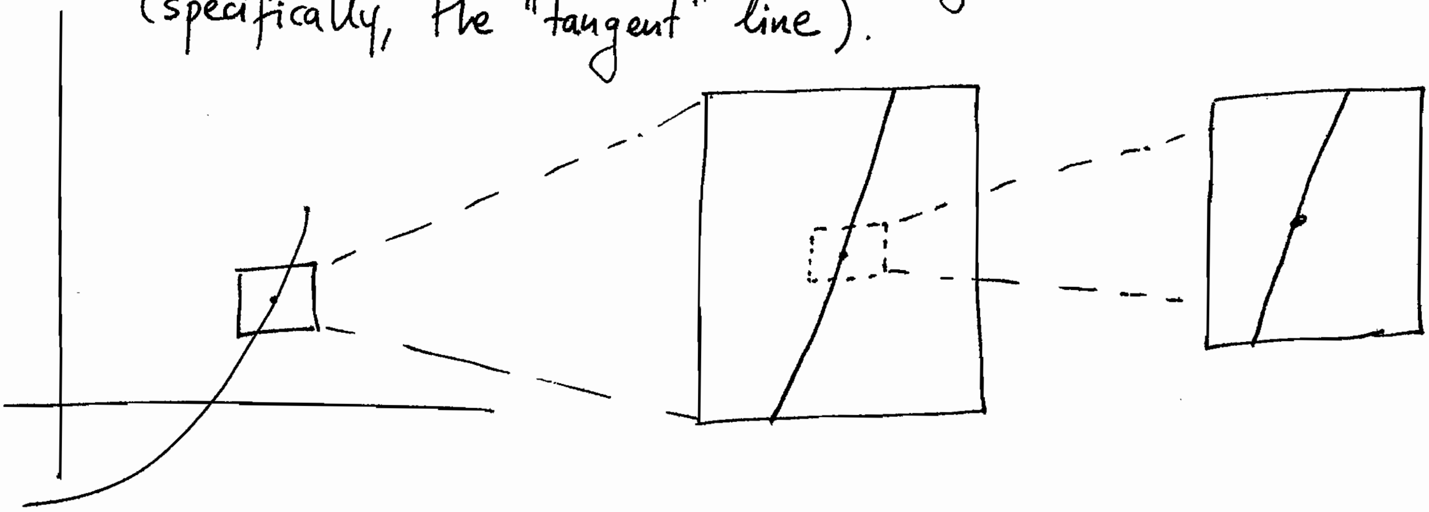
$$\underbrace{\left|\frac{x_{k+1} - \sqrt{a}}{\sqrt{a}}\right|}_{\text{relative error}} \leq C \underbrace{\left|\frac{x_k - \sqrt{a}}{\sqrt{a}}\right|}_{\text{relative error}}^2 \quad \text{for } k \geq k_0$$

Thus, subsequent iterations double the correct significant digits.

This example is a special case of an algorithm for solving nonlinear equations, known as Newton's method (or, the Newton-Raphson method)

Here is the general idea:

If we "zoom" close enough to any smooth function, it looks more and more like a straight line (specifically, the "tangent" line).

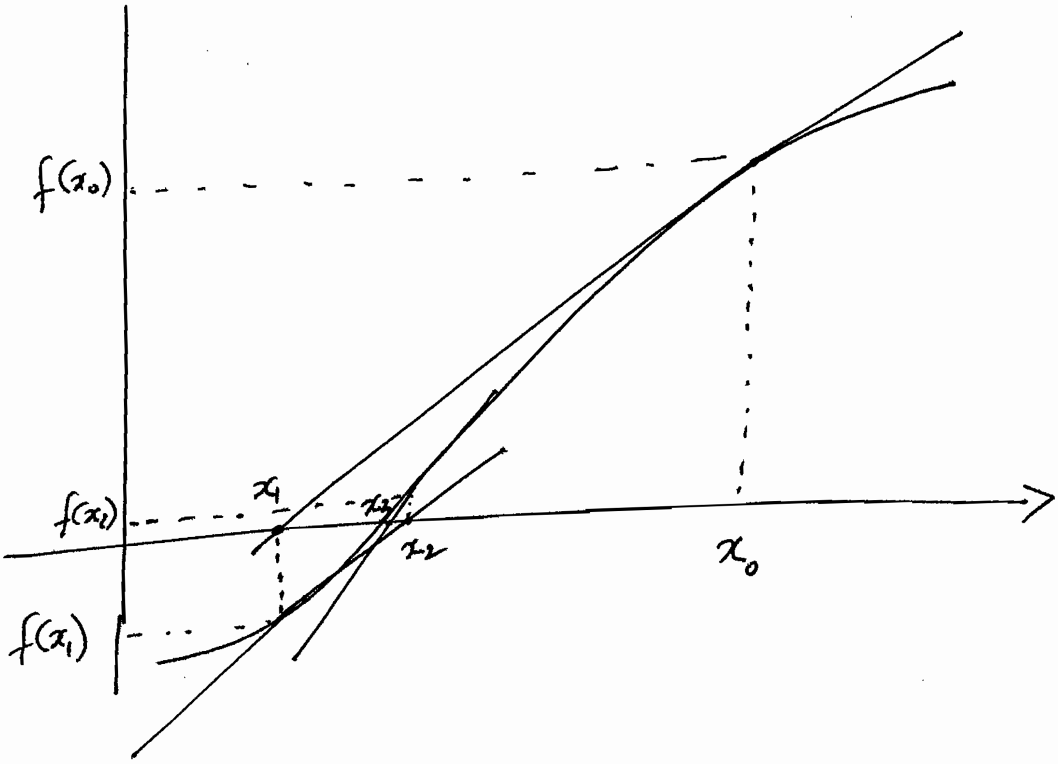


The newton method suggests:

If, after k iterations we have approximated the solution of $f(x)=0$ (a nonlinear equation) as x_k, then:

→ Form the tangent line at $(x_k, f(x_k))$

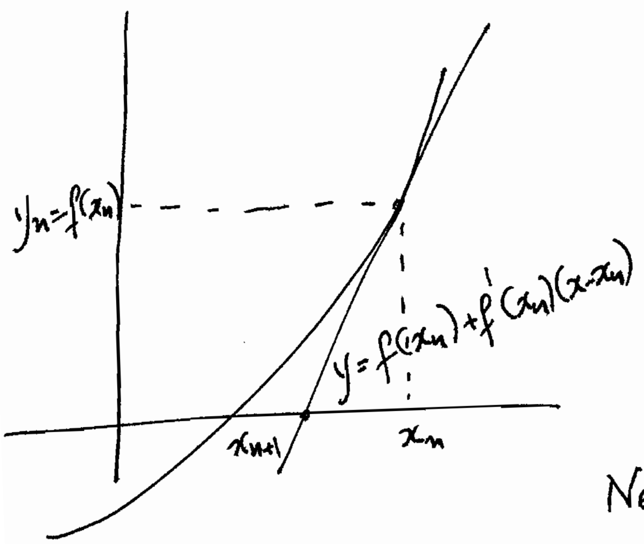
→ Select x_{k+1} as the intersection of the tangent line with the horizontal axis ($y=0$).



If $(x_n, y_n) = (x_n, f(x_n))$ the tangent line to the plot of $f(x)$ at x_n, y_n is

$$y - y_n = \lambda\,(x - x_n) \quad \text{where } \lambda = f'(x_n) \text{ is the slope}$$

thus $y = y_n + \lambda(x - x_n) \Rightarrow \boxed{y = f(x_n) + f'(x_n)(x - x_n)}$



If we set $y = 0$, we obtain

$$f(x_n) + f'(x_n)(x - x_n) = 0$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)} := x_{n+1}$$

Newton's method: $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$

Consider our previous example. The square root $\sqrt{a}$ is the solution to the nonlinear equation $f(x) = x^2 - a = 0$.

Newton's method gives

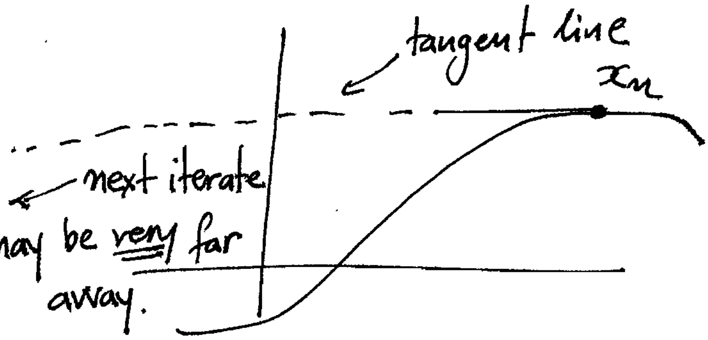
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 - x_n^2 + a}{2x_n}$$

$$\Longrightarrow \quad x_{n+1} = \frac{x_n^2 + a}{2x_n}$$ which is the method we considered previously.

A few comments about Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

* It requires the function $f(x)$ to be not only continuous, but differentiable, as well. We will later see variants that do not explicitly require knowledge of f' (in cases where the derivative may be hard to compute)

* If ever we have an approximation x_n with $f'(x_n) \approx 0$, we should expect problems, especially if we are not close to a solution (we will be nearly dividing by 0). Graphically:



* If we don't start close to a solution, convergence may not be guarranteed (or it may take a large number of iterations).

Fixed point iteration (§2.1)

Newton's method is a special case for a broader category of methods for solving nonlinear equations, called fixed point iteration methods

Generally, if $f(x)=0$ is the nonlinear equation we seek to solve, a fixed point iteration method proceeds as follows:

$$x_0 = \text{initial guess}$$

$$x_{n+1} = g(x_n)$$

where $g(x)$ is a properly designed function.

Thus, we construct the sequence: $x_0, x_1, x_2, \dots, x_k, \dots$ which should ideally converge to a solution of $f(x)=0$.

The following questions arise:

(i) If the iteration converges, does it converge to a solution of $f(x)=0$?

(ii) Is the iteration guaranteed to converge?

(iii) How fast does the iteration converge

(and (iv) When do we stop iterating?).

Q1. If $\{x_k\}$ converges, does it converge to a solution of $f(x)=0$?

Taking limits on $x_{n+1} = g(x_n)$, and assuming that $\lim x_n = a$ and g is continuous, we get

$$\lim x_{n+1} = \lim g(x_n)$$
$$a = g(a)$$

The simplest way to guarantee that a <u>is</u> a solution to $f(x)=0$ (i.e $f(a)=0$) is if we can show that

$a = g(a)$ & $f(a)=0$ are equivalent expressions.

There are more than one ways to make this happen, e.g:

$$f(x)=0 \iff x+f(x) = x \iff x = g(x), \text{ where } g(x)=x+f(x)$$

or $(x \neq 0)$
$$f(x)=0 \iff e^{-x}f(x)=0 \iff e^{-x}f(x)+x^2 = x^2 \iff$$
$$\iff \frac{e^{-x}f(x)+x^2}{x} = x \quad \text{or} \quad x=g(x),\ g(x) = \frac{e^{-x}f(x)+x^2}{x}$$

or $(f'(x)\neq 0)$
$$f(x)=0 \iff -\frac{f(x)}{f'(x)}=0 \iff x-\frac{f(x)}{f'(x)} = x \iff$$
$$\iff x=g(x) \text{ with } g(x) = x-\frac{f(x)}{f'(x)}$$ Newton's method!

Unfortunately, simply "constructing" $g(x)$ in a way that $x = g(x) \Leftrightarrow f(x) = 0$ does <u>not</u> imply that the iteration will converge! Consider $f(x) = x^2 - a = 0$ (solution: $\sqrt{a}$)

Simple formulations such as:

$$x^2 = a \Leftrightarrow x = \frac{a}{x} = g(x) \quad ?$$

$$x_1 = \frac{a}{x_0}, \quad x_2 = \frac{a}{x_1} = \frac{a}{(a/x_0)} = x_0$$

thus the sequence alternates forever $x_0, x_1, x_0, x_1, \ldots$

or:

$$\underbrace{x^2 - a + x}_{g(x)} = x$$

Let $x_0 = 3$, $a = 10$

$$x_1 = x_0^2 - 10 + x_0 = 9 - 10 + 3 = 2$$

$$x_2 = x_1^2 - 10 + x_1 = 4 - 10 + 2 = -4$$

$$x_3 = x_2^2 - 10 + x_2 = 16 - 10 - 4 = 2$$

repeats

or Let $x_0 = 5$ $a = 10$ (bad initial guess)

$$x_1 = 25 - 10 + 5 = 20$$

$$x_2 = 400 - 10 + 20 = 410$$

$\vdots$ diverges to $+\infty$!

It is even likely that for some iterations $x = g(x)$, the sequence will diverge <u>regardless</u> of how good the initial guess is (other than absolutely correct!)

Fortunately, there are ways to ensure $\{x_k\}$ converges, for certain choices of $g(x)$

<u>Definition</u>: A function $g(x)$ is a <u>contraction</u> in the interval $[a,b]$, if

$$|g(x)-g(y)| \leq L\,|x-y|$$

for any $x,y \in [a,b]$ and $L \in [0,1)$.

<u>examples</u>: $g(x) = \frac{x}{2}$: $|g(x)-g(y)| = \frac{1}{2}|x-y|$ for any x,y!

$g(x) = x^2$ in $[0.1, 0.2]$!

$$|g(x)-g(y)| = |x^2-y^2| = |x+y|\,|x-y| \leq .3\,|x-y|$$

in this case we really needed $x,y \in [a,b]$!

With a contraction, we can show the following:

* Let a be the real solution of $f(x)=0$, and assume $|x_0 - a| < \delta$ (δ = some positive number).
If g is a contraction on $(a-\delta, a+\delta)$, the fixed point iteration converges to a!

Proof: Since a is a solution, we have $g(a) = a$!

$$|x_1 - a| = |g(x_0) - g(a)| \leq L|x_0 - a| < L\delta$$

$$|x_2 - a| = |g(x_1) - g(a)| \leq L|x_1 - a| < L^2\delta$$

$$\vdots$$

$$|x_k - a| < L^k \delta$$

since $L < 1$ we have $\lim |x_k - a| = 0$

i.e. $x_k \to a$.

<u>Note</u>: For simplicity we assumed that a value <u>a</u> s.t. $g(a) = a$ <u>does exist</u>. However if g is a contraction we can more generally show that the fixed point iteration <u>will</u> converge, and the limit of course will satisfy $g(a) = a$.

In some cases, it can be a lot of work to show that g is a contraction, using the definition. However, if we can compute the derivative g' we have a simpler criterion:

* If g is differentiable, and $|g'(x)| \leq L$ (where $0 \leq L < 1$), in $[a,b]$, then g is a contraction on $[a,b]$.

Proof Let $x, y \in [a,b]$ (with $x \neq y$).

The mean value theorem states that

$$\frac{g(x)-g(y)}{x-y} = g'(\xi) \quad \text{for some } \xi \in (x,y)$$

if $|g'(\xi)| \leq L$, then

$$\left|\frac{g(x)-g(y)}{x-y}\right| \leq L \Rightarrow |g(x)-g(y)| \leq L|x-y|$$

examples: $g(x) = \sin\left(\frac{2x}{3}\right)$

$$|g'(x)| = \frac{2}{3}\left|\cos\left(\frac{2x}{3}\right)\right| \leq \frac{2}{3} < 1$$

Let's try to apply the derivative criterion, to see if

$g(x) = x - \frac{f(x)}{f'(x)}$ (from Newton's method) is a contraction!

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = 1 - 1 + \frac{f(x)f''(x)}{(f'(x))^2}$$

$$= \frac{f(x)f''(x)}{(f'(x))^2}$$

Now, let us assume that:

- $f(a) = 0$ (i.e a is a solution of $f(x)=0$)
- $f'(a) \neq 0$
- f'' is bounded near a (for example, if it is continuous).

then $$\lim_{x\to a} g'(x) = \frac{f(x)f''(x)}{(f'(x))^2} = 0$$

This means that there is an interval $(a-\delta, a+\delta)$ where

$|g'(x)|$ is <u>small</u> (since $\lim g'(x) = 0$!)

or specifically $|g'(x)| \leq L$ $(L < 1)$

This means that g is a contraction on $(a-\delta, a+\delta)$ and, if $x_0 \in (a-\delta, a+\delta)$, the iteration will converge to the solution $\underline{\underline{a}}$.