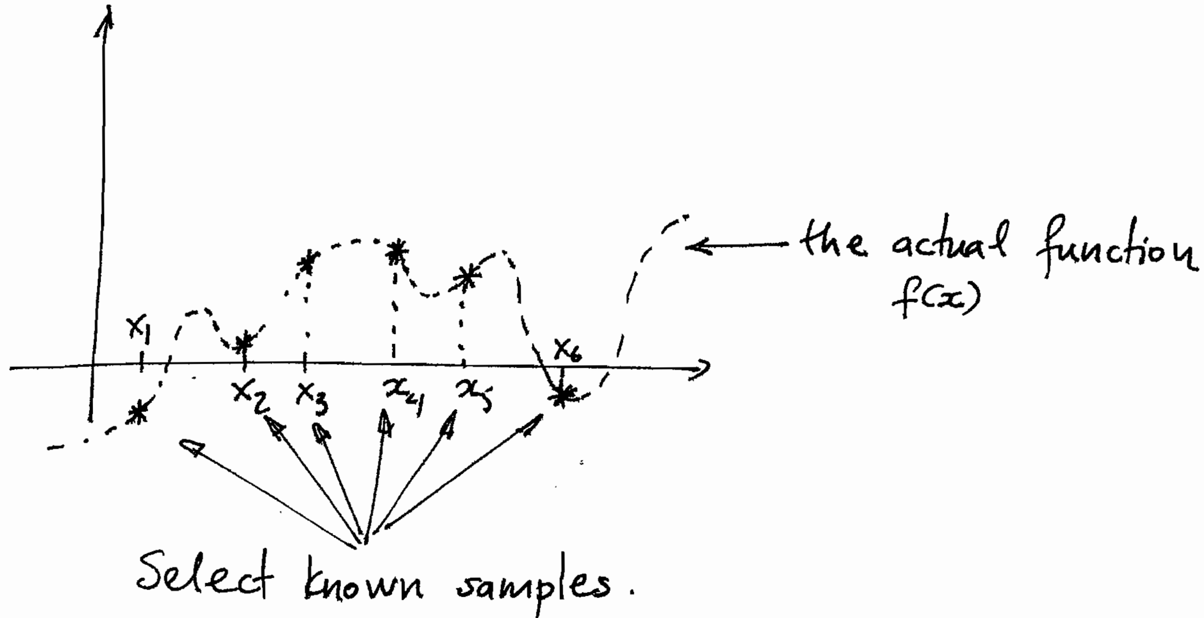
Interpolation (§4.2, 4.3)

We are often interested in a certain function $f(x)$, but despite the fact that f may be defined over a certain interval $[a,b]$ (or, over all reals) we only know its precise value at selected points $x_1, x_2, \ldots, x_N$



There may be several good reasons why we could only have a limited number of values for $f(x)$, instead of its entire graph:

- Perhaps we do not know of an analytic formula for $f(x)$, because it is the result of a complex process, e.g. a temperature / density / concentration / velocity value in a lab experiment. Instead, we use measurements to sample $f(x)$

- Or, perhaps we do have a formula for $f(x)$, but this formula is not so eary to evaluate. Consider e.g.:

$$f(x) = \sin x \quad \text{or} \quad f(x) = \ln x \quad \text{or} \quad f(x) = \int_0^x e^{-t^2} dx.$$

Perhaps evaluating $f(x)$ for such examples is a very expensive operation, and we want to allow for a faster way to determine a "crude approximation". In the days when computers were not as widespread, trigonometric tables were very popular, for example:

Angle	$\sin\theta$	$\cos\theta$
...	...	...
44°	0.695	0.719
45°	0.707	0.707
46°	0.719	0.695
47°	0.731	0.682
...	...	...

Interpolation methods attempt to answer questions about the value of $f(x)$ at points <u>other</u> ~~than~~ than the ones it was sampled at. i.e What is $f(x)$ (where $x \neq x_i$, $i \in 1, \dots, N$) ?

<u>example 1</u>: A driver is headed towards city X. At 12:00pm, a GPS device recorded his position as being 215 miles away from X. At 12:20pm & 12:40pm the device recorded distances of 195mi & 170mi respectively.

(a) what distance is expected at 1:10pm?

(b) what was the distance at 12:35pm?

example 2: For the trigonometric table above, one could ask what is a good approximation of $\sin(45.8^\circ)$?

The question of how to reconstruct a continuous function $f(x)$ that agrees with the sampled values at $x_1, \ldots, x_N$ is not a straightforward one, especially since there is more than 1 way to accomplish that. First, some notation

Denote the known value of $f(x)$ at $x = x_1$ as $y_1 (= f(x_1))$

at $x = x_2$ as $y_2 (= f(x_2))$

$\vdots$

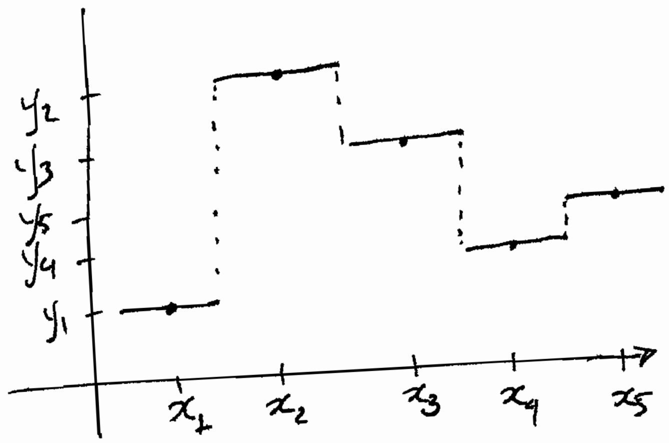
at $x = x_N$ as $y_N (= f(x_N))$

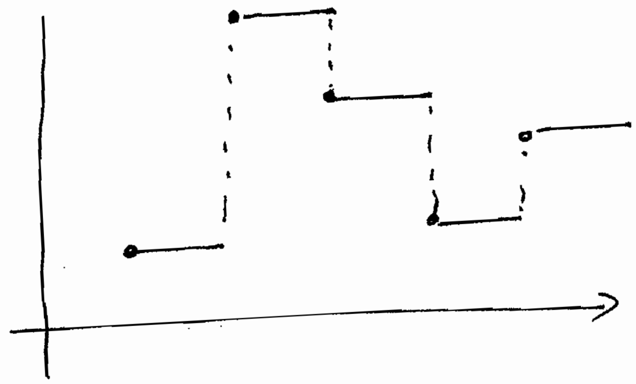
Graphically we seek to reconstruct a function $f(x)$, $x \in [a,b]$ such that the plot of f passes through all points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$.

Here are some possible ways to do that:

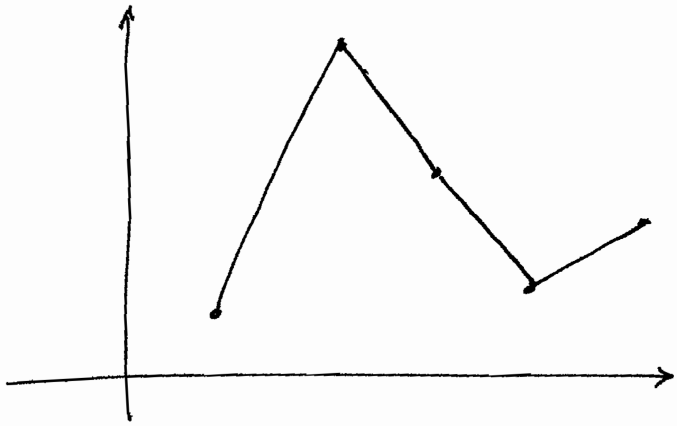
- For every x, pick the x_i closest to it, and set $f(x) = y_i$

or, just pick the value to the "left"





- Try connecting every 2 subsequent points with a straight line



- Or, try to find a smoother curve that connects them all

It is not trivial to argue that any particular one of these alternatives is "better", without having some knowledge of the nature of $f(x)$, or the purpose this reconstructed function will be used for. e.g.

- It may appear that the discontinuous approximation generated by the "pick the closest sample" method is awkward and not as well behaved. However, the real function $f(x)$ being sampled could have been just as discontinuous to begin with, e.g.

  $f(t)$ = the transaction amount for the customer of a bank being served at time $= t$.

- Sometimes, we may know that the real $f(x)$ is supposed to have some degree of smoothness, e.g. if $f(t)$ is the position of a moving vehicle along a highway, we would expect both $f(t)$ and $f'(t)$ (velocity), possibly even $f''(t)$ (acceleration) to be continuous functions of time.

  In this case, if we seek to estimate $f'(t)$ at a given time we may prefer the piecewise-linear reconstruction. If $f''(t)$ is needed, the even smoother method might be preferable.

Polynomial Interpolation

A commonly used approach is to use a properly crafted polynomial function

$$f(x) = P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

to interpolate the points $(x_0, y_0), \ldots, (x_k, y_k)$.

Some benefits:

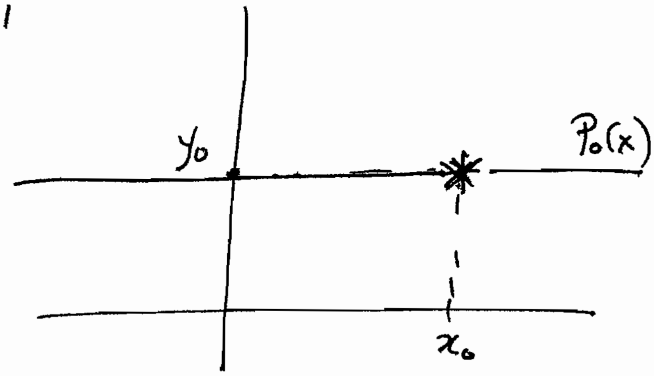
- Polynomials are relatively simple to evaluate (writing $P_n(x) = a_0 + x(a_1 + x(a_2 + x(a_3 \cdots + a_{n-1} + x a_n)))$ we see that one needs n multiplications & n additions)
- We can easily compute derivatives P_n', P_n'' if desired.
- Reasonably established procedure, to determine the a_i's.
- Polynomial approximations are familiar from, e.g. Taylor series.

And, some disadvantages:

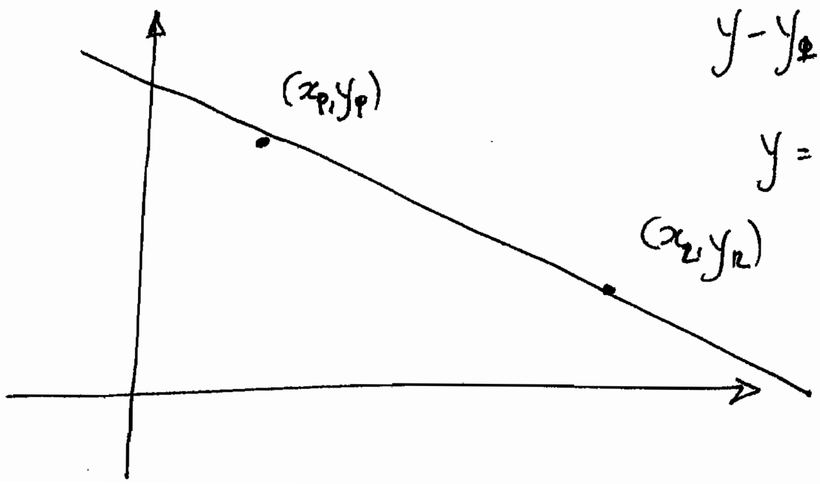
- Fitting polynomials can be problematic, when
  - (i) we have many data points (k is large), or
  - (ii) Some of the samples are too close together ($|x_i - x_j| =$ small)

In the interest of simplicity (and not only) we try to find the most basic, yet adequate, $P_n(x)$ that interpolates $(x_1,y_1),\dots,(x_k,y_k)$. Fore example.

- If $k=1$ (only one data sample) we have to interpolate through (x_1,y_1). A 0-degree polynomial (constant) will achieve that, if $a_0 = y_1$



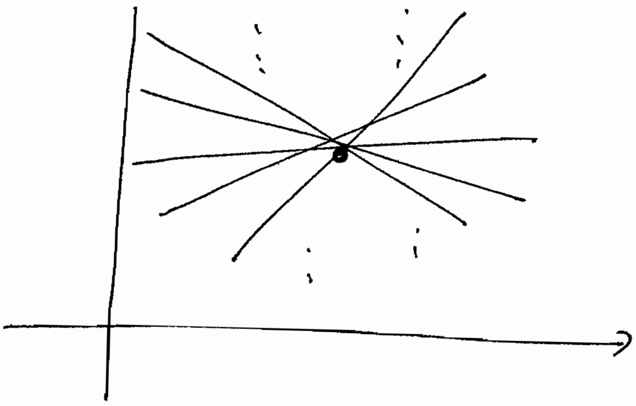
- If $k=2$ we have 2 points (x_1,y_1) & (x_2,y_2). A 0-degree polynomial $P_0(x)=a_0$ will not always be able to pass through both points (unless $y_1=y_2$), but a degree-1 polynomial $P_1(x)=a_0+a_1x$ always can



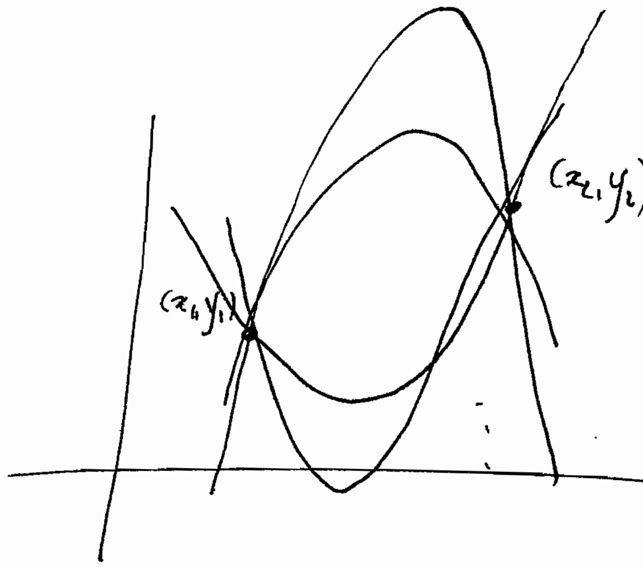
$$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y = \underbrace{y_1 - \frac{y_2-y_1}{x_2-x_1}x_1}_{a_0} + \underbrace{\frac{y_2-y_1}{x_2-x_1}}_{a_1} x$$

These are not the only polynomials that accomplish the task. e.g:



or



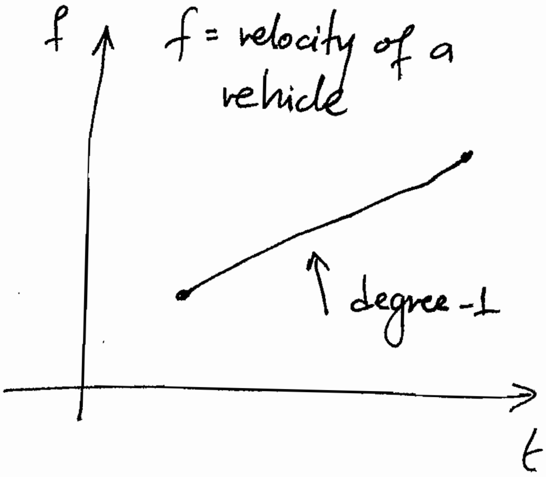
Interpolating (x_1, y_1) w/ 1-degree polynomials

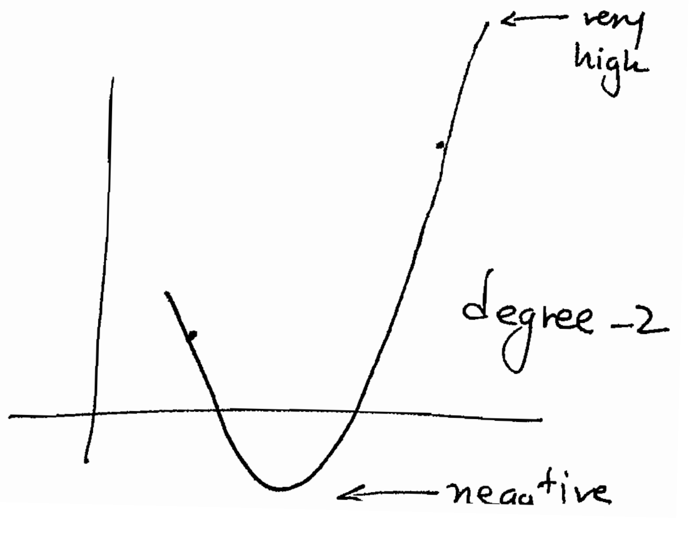
Using degree-2

The problem with using a degree higher than the minimum necessary is that:

- More than 1 solutions become available, with the "right" one being unclear
- Wildly varying curves become permissible, producing questionable approximations

e.g.





In fact we can show that using a polynomial $P_n(x)$ of degree $= n$ is the best choice when interpolating $n+1$ points. In this case the following properties are assured:

- Existence: Such a polynomial always exists (assuming that all the x_i's are different! it would be impossible for a function to pass through 2 points on the same vertical line). We will show this later, by constructing such a function.

- Uniqueness: We can sketch a proof:

  Assume that
$$P_n(x) = p_0 + p_1 x + \cdots + p_n x^n$$
$$Q_n(x) = q_0 + q_1 x + \cdots + q_n x^n$$
  both interpolate every (x_i, y_i), i.e. $P_n(x_i) = Q_n(x_i) = y_i \quad \forall i$

  Define another n-degree Polynomial
$$r_0 + r_1 x + \cdots + r_n x^n = R_n(x) = P_n(x) - Q_n(x).$$

  Apparently $R_n(x_i) = 0 \quad \forall i = 1, 2, \ldots, n+1.$

  From algebra we know that every polynomial of degree n has at most n real roots, unless it is the zero polynomial, i.e. $r_0 = r_1 = \cdots = r_n = 0$. Since we have $R_n(x) = 0$ for $n+1$ distinct ~~x~~ values, we must have $R_n(x) \equiv 0$
  or $P_n(x) \equiv Q_n(x)$.