CS412 Spring Semester 2011

Midterm #1 - Solutions to problems

Tuesday 8 March 2011

1. [30% = 5 questions × 6% each] MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). You do not need to provide a justification for your answer(s).

   (1) If the error e_k in a method for solving nonlinear equations satisfies the inequality $|e_{k+1}| \leq C|e_k|^d$, we say that the order of convergence is equal to d. Which of the following statements are true?
   (Circle or underline ALL correct answers)

   (a) When $d = 1$, the condition $C < 1$ is also necessary for convergence.

   (b) When $d = 2$, the condition $C < 1$ is also necessary for convergence.

   (c) If $d = 2$ and $C = 0.9$ the number of correct significant digits in our approximation will roughly double after each iteration.

   (2) Which of the following are true when comparing Newton's method to the Secant method for solving the equation $f(x) = 0$?
   (Circle or underline ALL correct answers)

   (a) The Secant method requires knowledge of the derivative $f'(x)$ while Newton's method does not require it.

   (b) An iteration of Newton's method is always computationally cheaper than an iteration of the Secant method.

   (c) The fact that the order of convergence is $d \approx 1.6$ for the Secant method, and $d = 2$ for Newton's method is *not* something we would consider a critical disadvantage for the Secant method.

   (3) When should we use Chebyshev points for polynomial interpolation?
   (Circle or underline the ONE most correct answer)

   (a) We should use them if we intend to use Lagrange interpolation.

   (b) We should use them if we have the flexibility to pick a specific set of x-values, and we know that both $f(x)$ and $f'(x)$ are bounded.

   (c) We should always use Chebyshev points, this is the best method.

(4) Which of the following can be claimed as advantages of the Vandermonde method for polynomial interpolation?
(Circle or underline ALL correct answers)

(a) Once the coefficients have been computed, evaluating either the polynomial or its derivative can be done very efficiently.

(b) It is easy to incrementally update the interpolant if we need to add one extra data point.

(c) Computing the coefficients with the Vandermonde method is more efficient than using divided differences.

(5) Which of the following are valid reasons for using piecewise polynomial interpolation, as opposed to using a single polynomial?
(Circle or underline ALL correct answers)

(a) Spurious oscillations associated with high-degree polynomials can be avoided by using lower-degree piecewise polynomials.

(b) Polynomial splines have well defined derivatives of any order.

(c) Piecewise polynomials can be extended to include more data points, while it is impossible to update a single polynomial interpolant incrementally to include additional points.

2. [20% = 4 questions × 5% each] SHORT ANSWER SECTION. Answer each of the following questions in no more than 1-2 sentences.

   (a) Write the equation that defines x_{k+1} as a function of x_k when using Newton's method to solve the nonlinear equation $x^2 = \sin(x)$.

   Answer: We can write the equation equivalently as $f(x) = 0$, where $f(x) = x^2 - \sin(x)$. In that case, Newton's method becomes:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - \sin(x_k)}{2x_k - \cos(x_k)}$$

   Note: It would also have been valid to set $f(x) = \sin(x) - x^2$, or $f(x) = 1 - \sin(x)/x^2$, or even $f(x) = x - \sin(x)/x$ etc. Setting $f(x) = \sqrt{\sin(x)}$ is *not* a valid choice, because we want the original equation to be equivalent to $f(x) = 0$, *not* $f(x) = x$.

   (b) In which situation does Newton's method exhibit linear (instead of quadratic) convergence?

   Answer: When $f'(a) = 0$ (where a is the root we are looking for). Equivalently, when the root has multiplicity more than 1.

   (c) Describe one of the benefits of using Chebyshev points for polynomial interpolation.

   Answer:

   - It ensures that the polynomial interpolant will converge to the function $f(x)$ being sampled as more data points are added, provided f and its first derivative are bounded.
   - It drastically reduces the risk of oscillatory interpolants associated with using high order polynomials

   (d) Describe a scenario when we would prefer using a standard cubic spline interpolation, rather than a Hermite spline.

   Answer:

   - When derivatives values at the data points are not known.
   - When a continuous second derivative is required of the reconstructed interpolant.

3. [15%] The *trisection* method is a modification of the bisection method for solving a nonlinear equation $f(x) = 0$. Its formal description is as follows

- Start with an interval $I_0 = [a, b]$ such that $f(a)f(b) < 0$.
- At the k-th step of the iteration we have $I_k = [a_k, b_k]$. Define:

$$c_0 = a_k, \quad c_1 = a_k + \frac{b_k - a_k}{3}, \quad c_2 = a_k + \frac{2(b_k - a_k)}{3}, \quad c_3 = b_k$$

  Let $j \in \{0, 1, 2\}$ be such that $f(c_j)f(c_{j+1}) < 0$ (such a j is guaranteed to exist). Then, define $I_{k+1} = [c_j, c_{j+1}]$ and continue with the iteration.
- After N iterations, the solution is approximated as $x \approx \frac{a_N + b_N}{2}$.

(a) What is the order of convergence of this method? A short qualitative explanation will suffice, you do not need to provide a formal proof.

(b) Would you consider this method to be a significant improvement over the Bisection method?

Solution

(a) The lengths of the intervals constructed by this method obey:

$$|I_{k+1}| \leq \frac{1}{3}|I_k|. \tag{1}$$

This indicates that trisection exhibits linear convergence.

(b) The convergence rate of the trisection method is linear, just as with bisection; the only difference is the factor $1/3$ in equation (1), which is $1/2$ for the bisection method. However, each step of the trisection method requires at least twice as many function evaluations as the bisection method; thus in the time necessary to do 1 iteration of trisection we could likely afford to do 2 iterations of bisection, reducing the error by a total factor of $1/4$. Thus, the trisection method reflects a net loss in accuracy per computational cost incurred.

4. [15%] Use Lagrange interpolation to find a cubic polynomial that interpolates the following four data points:

$$(-2,-1)$$
$$(-1,3)$$
$$(0,1)$$
$$(1,-1)$$

Reminder: Lagrange polynomials are given by the formula:

$$l_i(x) = \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)}$$

Solution

$$(-2,-1) = (x_0, y_0)$$
$$(-1,3) = (x_1, y_1)$$
$$(0,1) = (x_2, y_2)$$
$$(1,-1) = (x_3, y_3)$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x+1)x(x-1)}{(-2+1)(-2)(-2-1)} = -\frac{x^3-x}{6}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x+2)x(x-1)}{(-1+2)(-1)(-1-1)} = \frac{x^3+x^2-2x}{2}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+2)(x+1)(x-1)}{(0+2)(0+1)(0-1)} = -\frac{x^3+2x^2-x-2}{2}$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+2)(x+1)x}{(1+2)(1+1)1} = \frac{x^3+3x^2+2x}{6}$$

$$\begin{aligned} p(x) &= y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x) \\ &= -l_0(x) + 3l_1(x) + l_2(x) - l_3(x) \\ &= x^3 - 3x + 1 \end{aligned}$$

5. [20%] Using any of the methods we discussed in class, find a cubic polynomial $s(x)$, defined over $[0,1]$ that satisfies:

$$\begin{aligned} s(0) &= 2 \\ s'(0) &= -1 \\ s(1) &= 1 \\ s'(1) &= -3 \end{aligned}$$

Note: In case you decide to use the Hermite basis polynomials, those are given below:

$$\begin{aligned} h_{00}(x) &= 2x^3 - 3x^2 + 1 \\ h_{01}(x) &= -2x^3 + 3x^2 \\ h_{10}(x) &= x^3 - 2x^2 + x \\ h_{11}(x) &= x^3 - x^2 \end{aligned}$$

Solution

- Using basis polynomials:

$$\begin{aligned} p(x) &= s(0)h_{00}(x) + s(1)h_{01}(x) + s'(0)h_{10}(x) + s'(1)h_{11}(x) \\ &= 2(2x^3 - 3x^2 + 1) + 1(-2x^3 + 3x^2) + (-1)(x^3 - 2x^2 + x) + (-3)(x^3 - x^2) \\ &= -2x^3 + 2x^2 - x + 2 \end{aligned}$$

- Using divided differences:

x_0	$f[x_0]$			
x_0	$f[x_0]$	$f[x_0, x_0]$		
x_1	$f[x_1]$	$f[x_0, x_1]$	$f[x_0, x_0, x_1]$	
x_1	$f[x_1]$	$f[x_1, x_1]$	$f[x_0, x_1, x_1]$	$f[x_0, x_0, x_1, x_1]$

Populating the table according to the definitions of divided difference symbols (with or without repeated arguments), we get:

0	2			
0	2	-1		
1	1	-1	0	
1	1	-3	-2	-2

Ultimately:

$$s(x) = f[0] + f[0,0]x + f[0,0,1]x^2 + f[0,0,1,1]x^2(x-1) = -2x^3 + 2x^2 - x + 2$$

- It is also a valid solution to write $s(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and, consequently, $s'(x) = 3a_3x^2 + 2a_2x + a_1$ and proceed to solve the 4×4 system equivalent to the 4 conditions given above, to determine the unknown coefficients $a_3, \ldots, a_0$.