CS412 Introduction to Numerical Methods

Practice Midterm #1

1. MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). You do not need to provide a justification for your answer(s).

   (1) What would happen if we try to use an N-degree polynomial to interpolate N data points?
   (Circle or underline ALL correct answers)

   (a) This is the normal case. A unique polynomial can be determined.
   (b) Such an interpolant generally exists, but it is not unique.
   (c) A unique solution can still be determined, but it will be more oscillatory than using a $(N-1)$-degree polynomial.
   (d) Such an interpolant will generally not exist.

   (2) Which of the following are benefits of the secant method for solving $f(x) = 0$?
   (Circle or underline ALL correct answers)

   (a) It works without problems even near a solution $x = a$ that satisfies $f'(a) = 0$.
   (b) It does not require knowledge of the first derivative $f'(x)$.
   (c) It may be less computationally expensive than Newton's method, in cases where $f(x)$ is cheap to compute, but $f'(x)$ is much more expensive.
   (d) It achieves higher order of convergence than Newton's method.

   (3) Which of the following is NOT a valid reason for choosing piecewise cubic, instead of piecewise linear interpolation?
   (Circle or underline the ONE most correct answer)

   (a) Piecewise cubics can also achieve continuity of the derivatives of the interpolant.
   (b) Piecewise cubics generally converge faster than linear interpolants when increase the number of data points.
   (c) Piecewise cubics are cheaper to compute.
   (d) Piecewise cubics can predict derivative values better than linear interpolants.

2. SHORT ANSWER SECTION. Answer each of the following questions in no more than 1-2 sentences.

   (a) Describe one scenario where you would prefer using the Bisection method instead of Newton's method to solve an equation $f(x) = 0$.

   (b) Give an example of a function $g(x)$ which is *not* a contraction.

   (c) Are there some circumstances that may justify using the Vandermonde system approach instead of Lagrange interpolation to compute a polynomial interpolant for N given data points?

   (d) Name two reasons why we may prefer to use piecewise polynomial interpolation, rather than using a single polynomial interpolant with Chebyshev points.

   (e) Name two iterative methods for solving nonlinear equations that are *not* fixed point iterations.

   (f) Describe one plausible criterion for determining when to stop iterating Newton's method, when we try to find a solution to the nonlinear equation $f(x) = 0$.

3. Consider the fixed point iteration method $x_{k+1} = g(x_k)$ which we apply to find the solution to the nonlinear equation $f(x) = 0$. Now, consider the different fixed point iteration method, described by the recurrence:

$$x_{k+1} = h(x_k) = \frac{1}{2}\left[x_k + g(x_k)\right] \tag{1}$$

   (a) Show that if $g(x)$ is a contraction, then $h(x)$ is a contraction, too.

   (b) Show that if the iteration $x_{k+1} = g(x_k)$ converges linearly, then the iteration $x_{k+1} = h(x_k)$ will converge linearly as well.

   (c) Explain why even if the iteration $x_{k+1} = g(x_k)$ converges *quadratically*, the iteration $x_{k+1} = h(x_k)$ will generally have only linear convergence.

4. Consider the nonlinear equation $f(x) = (x-1)^3 = 0$. The only solution to this equation is obviously $x = 1$. Show that when we apply Newton's method to this equation:

   (a) The iteration always converges, and

   (b) The order of convergence is linear.

5. Use Newton interpolation (and divided differences) to determine the cubic polynomial that interpolates the following data points:

$$\begin{array}{c}(-1,-7)\\(0,-3)\\(1,-1)\\(2,5)\end{array}$$

6. In class, we have derived the following 2 bounds for the respective piecewise polynomial interpolation methods (assuming equally spaced data points, at intervals of size h):

   - For piecewise linear interpolation $|f(x) - s(x)| \leq \frac{1}{8}\|f''\|_\infty \cdot h^2$
   - For piecewise cubic interpolation $|f(x) - s(x)| \leq \frac{9}{384}\|f''''\|_\infty \cdot h^4$

   Consider the following 2 functions:

   (a) $f(x) = \sin(x), \quad x \in [0, 2\pi]$.

   (b) $f(x) = \sin(10x), \quad x \in [0, 2\pi]$.

   For each of these functions, explain if either the piecewise linear or the piecewise cubic interpolation will lead to a lower error. Is the piecewise cubic approach better under all circumstances?

7. Consider a piecewise *quadratic* polynomial function $s(x)$ that interpolates the N data points $(x_1, y_1), (x_1, y_1), \ldots, (x_N, y_N)$.

   In each subinterval $I_k = [x_k, x_{k+1}]$ we define $s(x)$ as a quadratic polynomial $s_k(x) = a_2^{(k)} x^2 + a_1^{(k)} x + a_0^{(k)}$, such that:

   - For $k = 1, 2, \ldots, n-2$, $s_k(x)$ should interpolate points (x_k, y_k), (x_{k+1}, y_{k+1}) and (x_{k+2}, y_{k+2}).
   - $s_{N-1}(x)$ should interpolate (x_{N-2}, y_{N-2}), (x_{N-1}, y_{N-1}) and (x_N, y_N).

   Derive a bound for the interpolation error $|f(x) - s(x)|$. You may assume that the length $h_k = |x_{k+1} - x_k|$ of each subinterval is constant and equal to h.

8. When using divided differences to compute Hermite splines, we defined a divided difference with two identical arguments $f[x_i, x_i]$ as the limit:

$$f[x_i, x_i] := \lim_{x_i^* \to x_i} f[x_i, x_i^*]$$

   and used this definition to show that $f[x_i, x_i] = f'(x_i) = y_i'$.

   (a) Let a divided difference $f[x_i, x_i, x_i]$ be defined as the limit:

$$f[x_i, x_i, x_i] := \lim_{\epsilon \to 0} f[x_i - \epsilon, x_i, x_i + \epsilon]$$

   Show that $f[x_i, x_i, x_i] = \frac{1}{2} f''(x_i) = \frac{1}{2} y_i''$.
   You may use (without proof) the following limit:

$$\lim_{\epsilon \to 0} \frac{f(x - \epsilon) - 2f(x) + f(x + \epsilon)}{\epsilon^2} = f''(x)$$

   (b) Explain how Newton interpolation and divided differences can be used to compute a cubic polynomial $s(x)$ over $[x_0, x_1]$ such that

$$\begin{aligned} s(x_0) &= y_0 \\ s'(x_0) &= y_0' \\ s''(x_0) &= y_0'' \\ s(x_1) &= y_1 \end{aligned}$$

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