# CS412 Spring Semester 2011

## Practice Midterm #2

1. MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). You do not need to provide a justification for your answer(s).

   (1) Numerical integration rules which use just one point per interval are:
   **(Circle or underline the ONE most correct answer)**

   (a) Always first order accurate. Methods that use 2 points are second order accurate and so on.

   (b) At most first order accurate, but can never have an order of accuracy of 2 or more.

   (c) Generally first order accurate, but sometimes could also be second order due to fortuitious cancellation.

   (2) A certain numerical integration rule integrates all cubic polynomials exactly. Which of the following property describes its accuracy?
   **(Circle or underline the ONE most correct answer)**

   (a) The method is exactly third order accurate.

   (b) The method is at least fourth order accurate.

   (c) The global error is proportional to $h^3$.

   (3) Which of the following methods are good choices for solving $\mathbf{Ax} = \mathbf{b}$, where $A$ is symmetric and positive definite?
   **(Circle or underline ALL correct answers)**

   (a) **LU** factorization with full pivoting.

   (b) **QR** factorization.

   (c) System of normal equations.

   (d) Gauss-Seidel method.

   (e) Jacobi method.

   (4) Why would we ever use an explcit method instead of an implicit one for solving an Initial Value Problem?
   **(Circle or underline ALL correct answers)**

   (a) Explicit methods are more robust for ODEs with unstable solutions.

   (b) Explicit methods do not require solving a nonlinear system.

   (c) If we are willing to use a small enough time step $dt$, each iteration of an explicit method is quite cheap.

   (d) Explicit methods are unconditionally stable.

2. SHORT ANSWER SECTION. Answer each of the following questions in no more than 1-2 sentences.

   (a) Describe one scenario where you would prefer using the **LU** factorization, instead of an iterative method such as Jacobi or Gauss-Seidel, for solving a linear system $\mathbf{Ax} = \mathbf{b}$.

   (b) What is one benefit of the **QR** factorization when compared to the normal equations, as methods for solving least squares problems?

   (c) Describe one reason why solving a system $\mathbf{Ax} = \mathbf{b}$ could be extremely challenging when $\mathbf{A}$ has a very high condition number.

   (d) Describe one valid reason for using Forward Euler, instead of Backward Euler to solve an initial value problem. Also, what would be a reason for choosing Backward Euler in this case?

   (e) Why is Simpson's rule potentially much more attractive than the trapezoidal rule, when approximating definite integrals?

   (f) Describe one plausible stopping criterion for determining when to stop an iterative solver for $\mathbf{Ax} = \mathbf{b}$, such as Jacobi or Gauss-Seidel.

3. Show the following properties:
   (a) $\|\mathbf{x}\|_2^2 = \mathbf{x}^T\mathbf{x}$ for any vector $\mathbf{x} \in \mathbf{R}$.
   (b) $\|\mathbf{Q}\|_2 = 1$, for any *orthogonal* matrix $\mathbf{Q}$.

4. Consider the $n > 3$ data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. We want to find a cubic polynomial $p(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ such that the graph of $p(x)$ approximates the given data points as much as possible. Write a Least Squares system $\mathbf{Ax} \approx \mathbf{b}$ which can be used to determine this cubic approximating polynomial. What does this system reduce to, in the case $n = 4$ ?

5. Show that the coefficient matrix in the system of normal equations (used for solving least squares problems) is always symmetric and positive definite.

6. Consider the $n \times n$ linear system $\mathbf{Ax} = \mathbf{b}$.
   (a) Show that if $\mathbf{A}$ is diagonal, the Jacobi method converges after just one iteration.
   (b) Show that if $\mathbf{A}$ is lower triangular, the Gauss-Seidel method converges after just one iteration.

7. The numerical integration rule known as *Simpson's 3/8 Rule* is defined as:

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

   (a) Determine the order of accuracy of this method.
   (b) Describe a composite rule based on the formula above.

8. The numerical integration rule known as *Milne's Rule* is defined as:

$$\int_a^b f(x)dx \approx \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]$$

   Determine the order of accuracy of this method.

9. Consider a numerical integration rule defined as:

$$\int_a^b f(x)dx \approx w_1 f(a) + w_2 f\left(\frac{2a+b}{3}\right) + w_3 f(b)$$

   where $w_1, w_2, w_3$ are undetermined constants. Find the values of $w_1, w_2, w_3$ such that this rule becomes 3rd order accurate.

10. Consider the ordinary differential equation

$$y'(t) = f(t, y). \tag{1}$$

   (a) If the solutions to equation (1) are stable, show that the solutions of

   $$z'(t) = f(t, z) + g(t) \tag{2}$$

   are also stable ($g(t)$ is an arbitrary function). Furthermore, if the solutions to equation (1) are asymptotically stable, show that so will be the solutions to the differential equation (2).

   (b) Consider the special case $f(t, z) = \lambda z$, $\lambda < 0$. Show that any function of the form

   $$z(t) = ce^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda \tau} g(\tau) d\tau$$

   is a solution of equation (2). If we assume that these are *all* the solutions to equation (2), can you show directly (without using the derivative criterion) that they are asymptotically stable?

11. We have seen that many 1-step methods for initial value problems are created by integrating the ODE as follows:

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y) d\tau$$

and then approximating the integral on the right hand side by a numerical integration rule.

Describe the method that results from using Simpson's rule for approximating this integral, and determine its stability condition on the model equation $y' = \lambda y,\ \lambda < 0$.